A note on Pricing, Duality and Symmetry for Two Dimensional Lévy Markets

José Fajardo[1] and Ernesto Mordecki[2]

[1] IBMEC Business School, Rio de Janeiro - Brazil
(e-mail: pepe@ibmecrj.br)
[2] Centro de Matemática, Facultad de Ciencias, Universidad de la República, Montevideo. Uruguay
(e-mail: ernesto.mordecki@gmail.com)

The aim of this work is to use a duality approach to study the pricing of derivatives depending on two stocks driven by a two dimensional Lévy process. The main idea is to apply Girsanov's Theorem for Lévy processes, in order to reduce the posed problem to the pricing of a one Lévy driven stock in an auxiliary market, baptized as the "dual market". In this way, we extend the results obtained by Gerber and Shiu [5] for two dimensional Brownian motion. Additionally we obtain a put-call relationship, that we call *duality*, and also a condition in order to have a *symmetry* condition in a Lévy market.

Key words: Lévy processes, Optimal stopping, Girsanov's Theorem, Dual Market Method, Derivative pricing, Symmetry

Mathematics Subject Classification (2000): 60G51, 91B28

JEL Classification Numbers: G12, G13

1 Introduction

Since Margrabe's 1978 paper [8], many important extensions have been carrying on to study derivatives written on two stocks. Margrabe studied the pricing of European options for the case of two non-dividend-paying stocks driven by a pair of Brownian motions, to be more exactly, the pricing of the right to change one asset for another at the end of some fixed period of time obtaining closed form formulas for this problem, extending in this way the Black and Scholes pricing model.

The American option pricing problem leads to the solution of an optimal stopping problem, that in general does not admit closed form solutions, even in the one asset case (see Jacka [6]). In the perpetual case, i.e. the option has no expiration date, Gerber and Shiu [5] obtain a closed form formula for Margrabe's and other related options using the optional sampling theorem, assuming that stock prices are driven by geometric Brownian motions, possible constantly correlated, and stocks pays constant rate continuous dividends.

In the present paper we consider the problem of pricing European and American type derivatives written on a two dimensional stock driven by a two dimensional Lévy processes (it can be said that the stock follows a *two dimensional geometric Lévy process*), with a payoff function homogeneous of an arbitrary degree. Additionally, the interest rate can also be stochastic, modeled by a third geometric Lévy process. As a related result, we obtain a relation between prices of call and put vanilla options in Lévy markets, and a condition on the jump structure of the process in order to obtain a *symmetric* Lévy market.

The paper is organized as follows: in section 2 we describe the market model and introduce the pricing problem. In section 3 we describe the *Dual Market Method*, which allows us to reduce the two stock problem with stochastic interest rate into a one stock problem with deterministic interest rate. In section 4 we study duality and symmetry in Lévy markets. A short conclusion is given in section 5.

2 Market Model

2.1 Multidimensional Lévy processes

Let $X = (X^1, \dots, X^d)$ be a d-dimensional Lévy process defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$. This means that X is a stochastically continuous stochastic process with independent increments, such that the distribution of $X_{t+s} - X_s$ does not depend on s, with $P(X_0 = 0) = 1$ and trajectories continuous from the left with limits from the right. The basis $\mathcal{B}$ is supposed to satisfy the usual assumptions, i.e. continuity from the right and $\mathcal{F}_0$ is P complete. For $z = (z_1, \dots, z_d)$ in $\mathbf{C}^d$, when the integral is convergent (and this is always the case if $z = i\lambda$ with λ in $\mathbf{R}^d$), Lévy-Khinchine formula states, that $\mathsf{E}e^{zX_t} = \exp(t\psi(z))$ where the function ψ is the *characteristic exponent* of the process, and is given by

$$\psi(z) = (a,z) + \frac{1}{2}(z,\Sigma z) + \int_{\mathbf{R}^d} \left(e^{(z,y)} - 1 - (z,y)\mathbf{1}_{\{|y|\le 1\}}\right)\Pi(dy), \tag{1}$$

where $a = (a_1,\dots,a_d)$ is a vector in $\mathbf{R}^d$, Π is a positive measure defined on $\mathbf{R}^d \setminus \{0\}$ such that $\int_{\mathbf{R}^d}(|y|^2 \wedge 1)\Pi(dy)$ is finite, and $\Sigma = ((s_{ij}))$ is a symmetric nonnegative definite matrix, that can always be written as $\Sigma = A'A$ (where $'$ denotes transposition) for some matrix A.

The triplet (a,Σ,Π) completely determines the law of the process X. Particular interest has the case when $\alpha = \int_{\mathbf{R}^d}\Pi(dy)$ is finite, i.e. X is a diffusion with jumps. Introducing F by $\Pi(dy) = \alpha F(dy)$, Lévy-Khinchine formula is (changing the value of a if necessary)

$$\psi(z) = (a,z) + \frac{1}{2}(z,\Sigma z) + \int_{\mathbf{R}^d} \left(e^{(z,y)} - 1\right)\Pi(dy), \tag{2}$$

and the process $X = \{X_t\}_{t\ge 0}$ can be represented by

$$X_t = at + AW_t + \sum_{k=1}^{N_t} Y_k,$$

where W is a standard d-dimensional Brownian motion, $N = \{N_t\}_{t\ge 0}$ is a Poisson process with parameter α, and $\{Y_k\}_{k\ge 1}$ is a sequence of independent d-dimensional random vectors with identical distribution $F(dy)$.

Another important case is when the coordinates of X are independent processes. This happens if and only if Σ is a diagonal matrix (and A can be chosen to be diagonal also) and the measure Π has support on the union of the coordinate axes, see E 12.10 in Sato [10]. In this case $\psi(z) = \sum_{k=1}^d \psi_k(z_k)$, where ψ_k is the characteristic exponent of the k-coordinate of X, given by

$$\psi_k(z_k) = a_k z_k + \frac{1}{2}s_{kk}z_k^2 + \int_{\mathbf{R}} \left(e^{z_k y} - 1 - z_k y\mathbf{1}_{\{|y|\le 1\}}\right)\Pi_k(dy),$$

where $\Pi_k(A) = \int_{\{x\in\mathbf{R}^d:\ x_k\in A\}}\Pi(dx)$.

2.2 Market and Problem

Consider a market model with three assets (S^1,S^2,S^3) given by

$$S_t^1 = e^{X_t^1}, \quad S_t^2 = S_0^2 e^{X_t^2}, \quad S_t^3 = S_0^3 e^{X_t^3} \tag{3}$$

where (X^1,X^2,X^3) is a three dimensional Lévy process, and for simplicity, and without loss of generality we take $S_0^1 = 1$. The first asset is the bond and is usually deterministic. Randomness in the bond $\{S^1_t\}_{t\geq 0}$ allows to consider more general situations, as for example the pricing problem of a derivative written in a foreign currency, referred as *Quanto option*.

Consider a function:

$$f\colon (0,\infty)\times(0,\infty)\to \mathbf{R}$$

homogeneous of an arbitrary degree α; i.e. for any $\lambda > 0$ and for all positive x, y

$$f(\lambda x, \lambda y) = \lambda^\alpha f(x, y).$$

In the above market a derivative contract with payoff given by

$$\Phi_t = f(S^2_t, S^3_t)$$

is introduced. Taking different homogeneous functions f we obtain several examples of options considered in the literature: Options to Default, Margrabe's Options, Swap Options, Quanto Options, and Equity-Linked Foreign Exchange Options. See the details in [3].

Assuming that we are under a risk neutral martingale measure, i.e. $\frac{S^k}{S^1}$ ($k = 2, 3$) are P-martingales (P is an equivalent martingale measure), we want to price the derivative contract just introduced. In the European case, the problem reduces to the computation of

$$E_T = E(S^2_0, S^3_0, T) = \mathsf{E}\left[e^{-X^1_T} f(S^2_0 e^{X^2_T}, S^3_0 e^{X^3_T})\right] \tag{4}$$

In the American case, if $\mathcal{M}_T$ denotes the class of stopping times up to time T, i.e:

$$\mathcal{M}_T = \{\tau:\ 0 \leq \tau \leq T,\ \tau \text{ stopping time}\}$$

for the finite horizon case, putting $T = \infty$ for the perpetual case, the problem of pricing the American type derivative introduced consists in solving an optimal stopping problem, more precisely, in finding the value function A_T and an optimal stopping time τ^* in $\mathcal{M}_T$ such that

$$\begin{aligned} A_T = A(S^2_0, S^3_0, T) &= \sup_{\tau\in\mathcal{M}_T} \mathsf{E}\left[e^{-X^1_\tau} f(S^2_0 e^{X^2_\tau}, S^3_0 e^{X^3_{\tau 3}})\right] \\ &= \mathsf{E}\left[e^{-X^1_{\tau^*}} f(S^2_0 e^{X^2_{\tau^*}}, S^3_0 e^{X^3_{\tau^*}})\right]. \end{aligned}$$

3 Dual Market method

The main idea to solve the posed problems is the following: make a change of measure through Girsanov's Theorem for Lévy processes, in order to reduce the original problems to a pricing problems for an auxiliary derivative written on one Lévy driven stock in an auxiliary market with deterministic interest rate. This method was used in Shepp and Shiryaev [11] and Kramkov and Mordecki [7] with the purpose of pricing American perpetual options with path dependent payoffs. It was employed by Araujo and Oliveira [1] to consider the pricing of swaps, and is strongly related with the election of the *numéraire* (see Geman et al. [4]). This auxiliary market will be called the *Dual Market.*

More precisely, observe that

$$e^{-X_t^1} f(S_0^2 e^{X_t^2}, S_0^3 e^{X_t^3}) = e^{-X_t^1+\alpha X_t^3} f(S_0^2 e^{X_t^2 - X_t^3}, S_0^3),$$

let $\rho = -\log \mathsf{E} e^{-X_1^1+\alpha X_1^3}$, that we assume finite. The process

$$Z_t = e^{-X_t^1+\alpha X_t^3+\rho t} \tag{1}$$

is a density process (i.e. a positive martingale starting at $Z_0 = 1$) that allow us to introduce a new measure, the *dual martingale measure* $\tilde{P}$, by its restrictions to each $\mathcal{F}_t$ by the formula

$$\frac{d\tilde{P}_t}{dP_t} = Z_t.$$

Denote now $\tilde{X}_t = X_t^2 - X_t^3$ and $S_t = S_0^2 e^{\tilde{X}_t}$. Finally, let

$$F(x) = f(x, S_0^3).$$

With the introduced notations, under the change of measure we obtain our main results:

$$E_T = \tilde{\mathsf{E}}\left[e^{-\rho T} F(S_T)\right], \qquad A_T = \sup_{\tau \in \mathcal{M}_T} \tilde{\mathsf{E}}\left[e^{-\rho\tau} F(S_\tau)\right]. \tag{2}$$

The concluding step to compute the prices in (2) is to determine the law of the process X under the auxiliary probability measure $\tilde{P}$, what is done in the following result, whose proof can be found in [3].

Lemma 1. *Let X be a Lévy process on $\mathbf{R}^d$ with characteristic exponent given in* (1). *Let u and v be vectors in $\mathbf{R}^d$. Assume that $\mathsf{E}e^{(u,X_1)}$ is finite, and denote $\rho = -\log \mathsf{E}e^{(u,X_1)} = -\psi(u)$. In this conditions, introduce the probability measure $\tilde{P}$ by its restrictions $\tilde{P}_t$ to each $\mathcal{F}_t$ by*

$$\frac{d\tilde{P}_t}{dP_t} = \exp[(u, X_t) + \rho t].$$

Then

(a) *the law of the unidimensional Lévy process $\{(v, X_t)\}_{t\geq 0}$ under $\tilde{P}$ is given by the triplet*

$$\begin{cases} \tilde{a} = (a, v) + \frac{1}{2}[(v, \Sigma u) + (u, \Sigma v)] + \int_{\mathbf{R}^d} e^{(u,y)}(v, y)\mathbf{1}_{\{|(v,y)|\leq 1, |x|>1\}}\Pi(dx) \\ \tilde{\sigma}^2 = (v, \Sigma v) \\ \tilde{\pi}(A) = \int_{\mathbf{R}^d} \mathbf{1}_{\{(v,y)\in A\}} e^{(u,y)}\Pi(dy). \end{cases} \tag{3}$$

(b) *In the particular case when X is a diffusion with jumps which characteristic exponent given in* (2) *the law of the unidimensional Lévy process $\{(v, X_t)\}_{t\geq 0}$ under $\tilde{P}$ is given by the triplet*

$$\begin{cases} \tilde{a} = (a, v) + \frac{1}{2}[(v, \Sigma u) + (u, \Sigma v)] \\ \tilde{\sigma}^2 = (v, \Sigma v) \\ \tilde{\pi}(A) = \int_{\mathbf{R}^d} \mathbf{1}_{\{(v,y)\in A\}} e^{(u,y)} \Pi(dy). \end{cases} \tag{4}$$

Furthermore, the intensity of the Poisson process under $\tilde{P}$ is given by

$$\tilde{\alpha} = \int_{\mathbf{R}^d} e^{(u,y)} \Pi(dy) = \alpha \int_{\mathbf{R}^d} e^{(u,y)} F(dy)$$

(c) *Assume* (b)*, and let $\Pi(dy) = \alpha F(dy)$ where F is the common distribution of the random variables $\{Y_k\}_{k\geq 1}$, and has characteristic function (under P) given by*

$$\phi(z) = \int_{\mathbf{R}^d} e^{(z,y)} F(dy).$$

Then, the characteristic function of the same random variables under $\tilde{P}$ is given by

$$\tilde{\phi}(\theta) = \frac{\phi(\theta v + u)}{\phi(u)}. \tag{5}$$

4 Put-Call Duality and Symmetry

In this section, relying on the same type of arguments of previous sections, we obtain a relationship between call and put vanilla options, that holds both in the European and in the American case, that we denominate *put-call duality*. Based on this relation, we obtain conditions to have *put-call symmetry*.

In order to do this, with the previous notations, consider $X_t^1 = rt,\ X_t^2 = 0$ and $X_t^3 = X_t$, with X_t a Lévy process. In other words we have a market with two assets, $B_t = e^{rt}$ and $S_t = S_0 e^{X_t},\ S_0 > 0$.

We also assume that the stock pays dividends, with constant rate $\delta \geq 0$, and as in section 2, that the probability measure P is the chosen equivalent martingale measure. In other words, prices are computed as expectations with respect to P, and the discounted and reinvested process $\{e^{-(r-\delta)t} S_t\}$ is a P–martingale.

Let us assume that τ is a stopping time with respect to the given filtration $\mathcal{F}$, that is $\tau\colon \Omega \to [0, \infty]$ belongs to $\mathcal{F}_t$ for all $t \geq 0$; and introduce the notation

$$\mathcal{C}(S_0, K, r, \delta, \tau, \psi) = \mathsf{E} e^{-r\tau}(S_\tau - K)^+ \tag{1}$$

$$\mathcal{P}(S_0, K, r, \delta, \tau, \psi) = \mathsf{E} e^{-r\tau}(K - S_\tau)^+ \tag{2}$$

If $\tau = T$, where T is a fixed constant time, then formulas (1) and (2) give the price of the European call and put options respectively.

Proposition 2 (Put-Call duality). *Consider a Lévy market with driving process X with characteristic exponent $\psi(z)$ given by*

$$\psi(q) = iaq - \frac{1}{2}\sigma^2 q^2 + \int_{\mathbf{R}} \big(e^{iqy} - 1 - iqy\mathbf{1}_{\{|y|<1\}}\big)\Pi(dy), \tag{3}$$

defined on the set

$$\mathbf{C}_0 = \Big\{z = p + iq \in \mathbf{C}\colon \int_{\{|y|>1\}} e^{py}\Pi(dy) < \infty\Big\}. \tag{4}$$

Then, for the expectations introduced in (1) *and* (2) *we have*

$$\mathcal{C}(S_0, K, r, \delta, \tau, \psi) = \mathcal{P}(K, S_0, \delta, r, \tau, \tilde{\psi}), \tag{5}$$

where

$$\tilde{\psi}(z) = \tilde{a}z + \frac{1}{2}\tilde{\sigma}^2 z^2 + \int_{\mathbf{R}} \big(e^{zy} - 1 - zy\mathbf{1}_{\{|y|\le 1\}}\big)\tilde{\Pi}(dy) \tag{6}$$

is a characteristic exponent (of a certain Lévy process) that satisfies

$$\tilde{\psi}(z) = \psi(1-z) - \psi(1), \quad \textit{for } 1 - z \in \mathbf{C}_0.$$

and in consequence,

$$\begin{cases} \tilde{a} & = \delta - r - \sigma^2/2 - \int_{\mathbf{R}} \big(e^y - 1 - y\mathbf{1}_{\{|y|\le 1\}}\big)\tilde{\Pi}(dy), \\ \tilde{\sigma} & = \sigma, \\ \tilde{\Pi}(dy) & = e^{-y}\Pi(-dy). \end{cases} \tag{7}$$

The proof of this proposition can be found in [3].

Observe that if we take a deterministic time $\tau = T$ in (5), we obtain that the price of an European call option in the risk neutral market (when X has a law characterized by ψ) coincides with the price of an European put option (with different parameters) in the dual market (when X has a law characterized by $\tilde{\psi}$).

As this relation holds with for any arbitrary stopping time, taking supremum in the class $\mathcal{M}_T$ we obtain that the same relation holds true for American options.

4.1 Symmetric markets

It is interesting to note, that in a market with no jumps the distribution (or laws) of the discounted (and reinvested) stocks in both the given and dual Lévy markets coincide. It is then natural to define a market to be *symmetric* when this relation hold, i.e. when

$$\mathcal{L}\big(e^{-(r-\delta)t+X_t} \mid P\big) = \mathcal{L}\big(e^{-(\delta-r)t-X_t} \mid \tilde{P}\big), \tag{8}$$

meaning equality in law. In view of (7), and to the fact that the characteristic triplet determines the law of a Lévy processes, we obtain that a necessary and sufficient condition for (8) to hold is

$$\Pi(dy) = e^{-y}\Pi(-dy). \tag{9}$$

This ensures $\tilde{\Pi} = \Pi$, and from this follows $a - (r - \delta) = \tilde{a} - (\delta - r)$, giving (8), as we always have $\tilde{\sigma} = \sigma$. Condition (9) answers a question raised by Carr and Chesney (1996), see [2].

5 Conclusions

In this paper we have extended the results obtained by Gerber and Shiu [5] for the bidimensional Geometric Brownian Motion to the case of bidimensional Geometric Lévy motion. We have shown that using the *Dual Market Method* it is possible to price many derivatives, with payoffs homogeneous of any degree, written in terms of two assets driven by geometric Lévy motions, in the European case and for the American perpetual case. Another important fact in this paper is the possibility of having a stochastic discount, this allow us to consider derivatives as quanto derivatives. As a related result, we obtained a relation between prices of call and put vanilla options in Lévy, markets, and obtained a condition on the jump structure of the process in order to obtain a *symmetric* Lévy market.

Acknowledgments. The first author thanks the comments of conference participants at The 2004 Winter Meeting of the Econometric Society and III World Congress of The Bachelier Finance Society. (The usual disclaimer applies.) The second author thanks Esko Valkeila and Youri M. Kabanov for support.

References

1. Araujo, A., Oliveira R.: On the Pricing of European and American Swaps Options, IMPA Preprint B 122 (1997).
2. Carr, P., Chesney, M.: American Put Call Symmetry, Preprint (1996)
3. Fajardo, J.; Mordecki, E.: Duality and derivative pricing with Lévy processes. Pre-Publicaciones de Matemática de la Universidad de la República, Montevideo, Pre-Mat 76 (2003).
4. Geman, H., El Karoui, N., and Rochet,J.: Changes of numéraire, changes of probability measure and option pricing, Journal of Applied Probability, **32**, 443–458 (1995)
5. Gerber, H. U., Shiu, E. S. W.: Martingale Approach to Pricing Perpetual American Options on Two Stocks. Mathematical Finance, **6**, 303–322 (1996).
6. Jacka, S.D.: Optimal stopping and the American put, Mathematical Finance, **1**, 1–14 (1991)

7. Kramkov, D. O., Mordecki, E.: An integral option, Theory Probability and Its. Applications, **39**, 162–172 (1994)
8. Margrabe, W.: The Value of an Option to Exchange One Asset for Another, Journal of Finance, **33**, 177–186 (1978)
9. Mordecki, E. Optimal stopping and perpetual options for Lévy processes. Finance and Stochastics. **VI**, 473–493 (2002)
10. Sato, K. : Lévy processes and infinitely divisible distributions. Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge (1999)
11. Shepp, L. A., Shiryaev, A. N.: A new look at the Russian option Theory of Probability and its. Applications, **39**, 103–119 (1995)