Duality and Derivative Pricing with Lévy Processes

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2004 North American Winter Meeting of the Econometric Society

Related Works

- Margrabe, W., (1978), "The Value of an Option to Exchange One Asset for Another", *J. Finance.*
- Gerber, H. and W. Shiu, (1996), "Martingale Approach to Pricing American Perpetual Options on Two Stocks", *Math. Finance.*

- Schroder, Mark. (1999), "Change of Numeraire for Pricing Futures, Forwards, and Options", *Review of Financial Studies.*
- Peskir, G., and A. N. Shiryaev, (2001), "A Note on The Call-Put Parity and a Call-Put Duality", *Theory Probab. Appl.*
- Detemple, J. (2001), "American Options: Symmetry Property", Cambridge University Press.
- Carr, P. and Chesney, M. (1998), "American Put Call Symmetry", Morgan Stanley working paper.
- Carr, P., Ellis, K., and Gupta, V., (1998), "Static Hedging of Exotic Options", *J. Finance.*

Some Relevant Derivatives

- Margrabe's Options:

  a) $f(x,y) = \max\{x,y\}$ *Maximum Option*

  b) $f(x,y) = |x-y|$ *Symmetric Option*

  c) $f(x,y) = \min\{(x-y)^+, ky\}$ *Option with Proportional Cap*

- Swap Options

  $$f(x,y) = (x-y)^+$$

  Option to exchange one risk asset for another.

- Quanto Options

  $$f(x,y) = (x-ky)^+$$

  x is the foreign stock in foreign currency. Then we have the price of an option to exchange one foreign currency for another.

- Equity-Linked Foreign Exchange Option (ELF-X Option). Take

$$S: \text{ foreign stock in foreign currency}$$

and Q is the spot exchange rate. We use foreign market risk measure, then an ELF-X is an investment that combines a currency option with an equity forward. The owner has the option to buy S_t with domestic currency which can be converted from foreign currency using a previously stipulated strike exchange rate R (domestic currency/foreign currency). The payoff is:

$$\Phi_t = S_T(1 - RQ_T)^+$$

Then $f(x, y) = (y - Rx)^+$.

- Vanilla Options.

$$f(x, y) = (x - ky)^+$$

and

$$f(x, y) = (ky - x)^+.$$

Multidimensional Lévy Processes

Let $\mathbf{X} = (X^1, \ldots, X^d)$ be a d-dimensional Lévy process defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$.

This means that $\mathbf{X}$ stochastic process with independent increments, such that the distribution of $\mathbf{X}_{t+s} - \mathbf{X}_s$ does not depend on s, with $P(\mathbf{X}_0 = 0) = 1$ and trajectories continuous from the left with limits from the right.

For $z = (z_1, \ldots, z_d)$ in C^d, when the integral is convergent (and this is always the case if $z = i\lambda$ with λ in R^d, Lévy-Khinchine formula states, that

$$\mathsf{E}e^{zX_t} = \exp(t\Psi(z))$$

where the function Ψ is the *characteristic exponent* of the process, and is given by

$$\Psi(z) = (a, z) + \frac{1}{2}(z, \Sigma z) + \int_{R^d} \left(e^{(z,y)} - 1 - (z, y)\mathbf{1}_{\{|y|\leq 1\}}\right)\Pi(dy), \tag{1}$$

where $a = (a_1, \ldots, a_d)$ is a vector in $\mathbb{R}^d$, Π is a positive measure defined on $R^d \setminus \{0\}$ such that $\int_{R^d}(|y|^2 \wedge 1)\Pi(dy)$ is finite, and $\Sigma = ((s_{ij}))$ is a symmetric nonnegative definite matrix, that can always be written as $\Sigma = A'A$ (where $'$ denotes transposition) for some matrix A.

Market Model and Problem

Consider a market model with three assets (S^1, S^2, S^3) given by

$$S_t^1 = e^{X_t^1}, \quad S_t^2 = S_0^2 e^{X_t^2}, \quad S_t^3 = S_0^3 e^{X_t^3} \tag{2}$$

where (X^1, X^2, X^3) is a three dimensional Lévy process, and for simplicity, and without loss of generality we take $S_0^1 = 1$. The first asset is the bond and is usually deterministic. Randomness in the bond $\{S_t^1\}_{t\geq 0}$ allows to consider more general situations, as for example the pricing problem of a derivative written in a foreign currency, referred as *quanto option*.
Consider a function:

$$f\colon (0,\infty) \times (0,\infty) \to I\!R$$

homogenous of an arbitrary degree α; i.e. for any $\lambda > 0$ and for all positive x, y

$$f(\lambda x, \lambda y) = \lambda^\alpha f(x,y).$$

In the above market a derivative contract with payoff given by

$$\Phi_t = f(S_t^2, S_t^3)$$

is introduced.

Assuming that we are under a risk neutral martingale measure, thats to say, $\frac{S^k}{S^1}$ $(k = 2, 3)$ are P-martingales, we want to price the derivative contract just introduced. In the European case, the problem reduces to the computation of

$$E_T = E(S_0^2, S_0^3, T) = \mathsf{E}\left[e^{-X_T^1} f(S_0^2 e^{X_T^2}, S_0^3 e^{X_T^3})\right] \tag{3}$$

In the American case, if $\mathcal{M}_T$ denotes the class of stopping times up to time T, i.e:

$$\mathcal{M}_T = \{\tau : \ 0 \leq \tau \leq T, \ \tau \text{ stopping time}\}$$

for the finite horizon case, putting $T = \infty$ for the perpetual case, the problem of pricing the American type derivative introduced consists in solving an optimal stopping problem, more precisely, in finding the value function A_T and an optimal stopping time τ^* in $\mathcal{M}_T$ such that

$$\begin{aligned} A_T = A(S_0^2, S_0^3, T) &= \sup_{\tau \in \mathcal{M}_T} \mathsf{E}\left[e^{-X_\tau^1} f(S_0^2 e^{X_\tau^2}, S_0^3 e^{X_{\tau^3}^3})\right] \\ &= \mathsf{E}\left[e^{-X_{\tau^*}^1} f(S_0^2 e^{X_{\tau^*}^2}, S_0^3 e^{X_{\tau^*}^3})\right]. \end{aligned}$$

How to obtain an EMM

Define

$$M(z,t;h) = \frac{M(z+h,t)}{M(h,t)}$$

where $M(h,t) = \mathsf{E}(e^{h\cdot X_t'})$. Now we will find a vector h^* such that the probability $dQ_t = \frac{e^{h^*\cdot X_t'}}{\mathsf{E}(e^{h^*\cdot X_t'})} dP_t$ be an EMM, in other words:

$$S_0^j = \mathsf{E}^*(e^{-rt} S_t^j) \ \ \forall j, \ \forall t$$

take $1_j = (0, \ldots, \underbrace{1}_{j-position}, \ldots, 0)$,then

$$r = \log[M(1_j, 1; h^*)] = \log\left[\frac{M(1_j + h^*, 1)}{M(h^*, 1)}\right]$$

Now in our model we need that $\{\frac{S_t^j}{S_t^1}\}$ be martingale, as $S_0^1 = 1$, then

$$S_0^j = \mathsf{E}^*(\frac{S_t^j}{S_t^1})$$

$$1 = \mathsf{E}^*(e^{X_t^j - X_t^1})$$

Defining $\bar{1}_j = (-1, 0, \ldots, \underbrace{1}_{j-position}, \ldots, 0)$, we have

$$1 = M(\bar{1}_j, 1; h^*)$$

observe that

$$e^{-X_t^1} f(S_0^2 e^{X_t^2}, S_0^3 e^{X_t^3}) = e^{-X_t^1 + \alpha X_t^3} f(S_0^2 e^{X_t^2 - X_t^3}, S_0^3).$$

Let $\rho = -\log \mathsf{E} e^{-X_1^1 + \alpha X_1^3}$, that we assume finite. The process

$$Z_t = e^{-X_t^1 + \alpha X_t^3 + \rho t}$$

is a density process (i.e. a positive martingale starting at $Z_0 = 1$) that allow us to introduce a new measure $\tilde{P}$ by its restrictions to each $\mathcal{F}_t$ by the formula

$$\frac{d\tilde{P}_t}{dP_t} = Z_t.$$

Denote now by $X_t = X_t^2 - X_t^3$, and $S_t = S_0^2 e^{X_t}$. Finally, let

$$F(x) = f(x, S_0^3).$$

With the introduced notations, under the change of measure we obtain

$$E_T = \tilde{\mathsf{E}} \left[e^{-\rho T} F(S_T) \right]$$
$$A_T = \sup_{\tau \in \mathcal{M}_T} \tilde{\mathsf{E}} \left[e^{-\rho \tau} F(S_\tau) \right]$$

The following step is to determine the law of the process X under the auxiliar probability measure $\tilde{P}$.

Lemma 1 *Let X be a Lévy process on R^d with characteristic exponent given in (1). Let u and v be vectors in R^d. Assume that $\mathsf{E}e^{(u,X_1)}$ is finite, and denote $\rho = \log \mathsf{E}e^{(u,X_1)} = \Psi(u)$. In this conditions, introduce the probability measure $\tilde{P}$ by its restrictions $\tilde{P}_t$ to each $\mathcal{F}_t$ by*

$$\frac{d\tilde{P}_t}{dP_t} = \exp[(u, X_t) - \rho t].$$

Then the law of the unidimensional Lévy process $\{(v, X_t)\}_{t \geq 0}$ under $\tilde{P}$ is given by the triplet

$$\begin{cases} \tilde{a} = (a, v) + \frac{1}{2}[(v, \Sigma u) + (u, \Sigma v)] + \int_{R^d} e^{(u,y)}(v, y)\mathbf{1}_{\{|(v,y)| \leq 1, |x|>1\}}\Pi(dx) \\ \tilde{\sigma}^2 = (v, \Sigma v) \\ \tilde{\pi}(A) = \int_{R^d} \mathbf{1}_{\{(v,y)\in A\}} e^{(u,y)}\Pi(dy). \end{cases} \tag{4}$$

Closed Formulae

European derivative

Let S_T^2 and S_T^3 be two risky assets, a contract with payoff $(S_T^2 - S_T^3)^+$ can be priced using *The Dual Market Method*:

$$D = \mathsf{E}\left[e^{-rT}(S_T^2 - S_T^3)^+\right].$$

$$= \int_{\mathcal{A}} e^{-rT}(S_0^2 e^{X_T^2} - S_0^3 e^{X_T^3})dP$$

Assuming for simplicity $S_0^2 = S_0^3 = 1$,
Then $\mathcal{A} = \{\omega \in \Omega : X_T^2(\omega) > X_T^3(\omega)\}$, we apply the method:

$$D = \int_{\mathcal{A}} e^{-rT}(e^{X_T^2} - e^{X_T^3})dP$$

$$= \int_{\{S_T > 1\}} e^{-rT} e^{X_T^3}(S_T - 1)dP$$

where $S_T = e^{X_T}$ and $X = X^2 - X^3$. Now the dual measure: $\rho = -\log \mathsf{E} e^{-r+X_1^3} = r - \log \mathsf{E} e^{X_1^3}$, then:

$$d\widetilde{P} = \frac{e^{X_T^3}}{\mathsf{E} e^{X_T^3}} dP$$

With all this:

$$D = e^{-\rho T} \int_{\{S_T > 1\}} (S_T - 1) d\widetilde{P}$$

$$D = e^{-\rho T} \int_{\{S_T > 1\}} S_T d\widetilde{P} - e^{-\rho T} \int_{\{S_T > 1\}} d\widetilde{P}$$

To reduce this expression we need a distribution for X under P, then applying the Lemma 1 we obtain the density of S_T under $\widetilde{P}$.

It is worth noting that instead of using a distribution for X, we can make the following change of measure:

$$d\widehat{P} = \frac{e^{X_T^2}}{\mathsf{E}e^{X_T^2}} dP$$

we obtain:

$$D = e^{-\rho T} \int_{\{S_T > 1\}} e^{X_T^2 - X_T^3} \frac{e^{X_T^3}}{\mathsf{E}e^{X_T^3}} dP - e^{-\rho T} \widetilde{P}(S_T > 1)$$

$$D = e^{-\rho T} \frac{\mathsf{E}e^{X_T^2}}{\mathsf{E}e^{X_T^3}} \widehat{P}(S_T > 1) - e^{-\rho T} \widetilde{P}(S_T > 1)$$

American Perpetual Swap

Proposition 1 *Let* $M = \sup_{o \leq t \leq \tau} X_t$ *with* τ *an independent exponential random variable with parameter* ρ*, then* $\tilde{\mathsf{E}} e^M < \infty$ *and*

$$A(S_0^2, S_0^3) = \frac{\tilde{\mathsf{E}}\left[S_0^2 e^M - S_0^3 \tilde{\mathsf{E}}(e^M)\right]}{\tilde{\mathsf{E}}(e^M)}$$

the optimal stopping time is

$$\tau_c^* = \inf\{t \geq 0, S_t \geq S_0^3 \tilde{\mathsf{E}}(e^M)\}$$

Put-Call Duality

Now consider a *Lévy market* with only two assets:

$$B_t = e^{rt}, \qquad r \geq 0,$$

$$S_t = S_0 e^{X_t}, \qquad S_0 = e^x > 0. \tag{5}$$

In this section we assume that the stock pays dividends, with constant rate $\delta \geq 0$.

In terms of the characteristic exponent of the process this means that

$$\psi(1) = r - \delta, \tag{6}$$

based on the fact, $\mathsf{E}e^{-(r-\delta)t+X_t} = e^{-t(r-\delta+\psi(1))} = 1$, condition (6) can also be formulated in terms of the characteristic triplet of the process X as

$$a = r - \delta - \sigma^2/2 - \int_{\mathbb{R}} \left(e^y - 1 - y\mathbf{1}_{\{|y|\leq 1\}}\right)\Pi(dy). \tag{7}$$

Now define the following set

$$\mathbb{C}_0 = \left\{ z = p + iq \in \mathbb{C} \colon \int_{\{|y|>1\}} e^{py}\Pi(dy) < \infty \right\}. \quad (8)$$

The set $\mathbb{C}_0$ consists of all complex numbers $z = p + iq$ such that $\mathsf{E}e^{pX_t} < \infty$ for some $t > 0$.

Now we consider call and put options, of both European and American types.

Let us assume that τ is a stopping time with respect to the given filtration $\mathcal{F}$, that is $\tau \colon \Omega \to [0, \infty]$ belongs to $\mathcal{F}_t$ for all $t \geq 0$; and introduce the notation

$$\mathcal{C}(S_0, K, r, \delta, \tau, \psi) = \mathsf{E}e^{-r\tau}(S_\tau - K)^+ \quad (9)$$

$$\mathcal{P}(S_0, K, r, \delta, \tau, \psi) = \mathsf{E}e^{-r\tau}(K - S_\tau)^+ \quad (10)$$

If $\tau = T$, where T is a fixed constant time, then formulas (9) and (10) give the price of the European call and put options respectively.

Dual Market

To prove our main result we need the following auxiliary market model:

$$B_t = e^{\delta t}, \qquad \delta \geq 0,$$

and a stock $\tilde{S} = \{\tilde{S}_t\}_{t\geq 0}$, modeled by

$$\tilde{S}_t = K e^{\tilde{X}_t}, \qquad S_0 = e^x > 0,$$

where $\tilde{X} = \{\tilde{X}_t\}_{t\geq 0}$ is a Lévy process with characteristic exponent under $\tilde{P}$ given by $\tilde{\psi}$ in (12). We call it *dual market.*

Proposition 2 *Consider a Lévy market with driving process X with characteristic exponent $\psi(z)$, defined on the set $\mathbb{C}_0$ in* (8). *Then, for the expectations introduced in* (9) *and* (10) *we have*

$$\mathcal{C}(S_0, K, r, \delta, \tau, \psi) = \mathcal{P}(K, S_0, \delta, r, \tau, \tilde{\psi}), \tag{11}$$

where

$$\tilde{\psi}(z) = \tilde{a}z + \frac{1}{2}\tilde{\sigma}^2 z^2 + \int_{\mathbb{R}} \left(e^{zy} - 1 - zy\mathbf{1}_{\{|y|\leq 1\}}\right)\tilde{\Pi}(dy) \tag{12}$$

is the characteristic exponent (of a certain Lévy process) that satisfies

$$\tilde{\psi}(z) = \psi(1 - z) - \psi(1), \quad \text{for } 1 - z \in \mathbb{C}_0,$$

and in consequence,

$$\begin{cases} \tilde{a} & = \delta - r - \sigma^2/2 - \int_{\mathbb{R}} \left(e^y - 1 - y\mathbf{1}_{\{|y|\leq 1\}}\right)\tilde{\Pi}(dy), \\ \tilde{\sigma} & = \sigma, \\ \tilde{\Pi}(dy) & = e^{-y}\Pi(-dy). \end{cases} \tag{13}$$

Proof:

We introduce the *dual martingale measure* $\tilde{P}$ given by its restrictions $\tilde{P}_t$ to $\mathcal{F}_t$ by

$$\frac{d\tilde{P}_t}{dP_t} = Z_t = e^{X_t-(r-\delta)t}, \quad t \geq 0,$$

Now

$$\begin{aligned} \mathcal{C}(S_0, K, r, \delta, \tau, \psi) &= \mathsf{E}e^{-r\tau}(S_0 e^{X_\tau} - K)^+ \\ &= \mathsf{E}Z_\tau e^{-\delta\tau}(S_0 - Ke^{-X_\tau})^+ \\ &= \tilde{\mathsf{E}}e^{-\delta\tau}(S_0 - Ke^{\tilde{X}_\tau})^+. \end{aligned}$$

where $\tilde{\mathsf{E}}$ denotes expectation with respect to $\tilde{P}$, and the process $\tilde{X} = \{\tilde{X}_t\}_{t\geq 0}$ given by $\tilde{X}_t = -X_t$ ($t \geq 0$) is the *dual process*.

To conclude the proof we must verify that the dual process $\tilde{X}$ is a Lévy process with characteristic exponent defined by (12) and (13).□

Remarks

- Our Proposition 2 is very similar to Proposition 1 in Schroder (1999). The main difference is that the particular structure of the underlying process (Lévy process is a particular case of the model considered by Schroder) allows to completely characterize the distribution of the dual process $\tilde{X}$ under the dual martingale measure $\tilde{P}$, and to give a simpler proof.

- It must be noticed that Peskir and Shiryaev (2001) propose the same denomination for a different relation in the Geometric Brownian Motion context:

$$(K - S_T)^+ = ((-S_T) - (-K))^+$$

Denoting $\tilde{S}_T = -S_T$, $\tilde{K} = -K$, $\tilde{\sigma} = -\sigma$ and introduzing $\widetilde{W}_t = -W_t$, we have

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

implies

$$d\tilde{S}_t = \mu \tilde{S}_t dt + \tilde{\sigma} \tilde{S}_t d\tilde{W}_t$$

Then

$$P_T(S_0, K; \sigma) = C_T(-S_0, -K; -\sigma)$$

Symmetric Markets

We say that a market is *symmetric* when

$$\mathcal{L}\big(e^{-(r-\delta)t+X_t} \mid P\big) = \mathcal{L}\big(e^{-(\delta-r)t-X_t} \mid \tilde{P}\big), \qquad (14)$$

meaning equality in law. In view of (13), and to the fact that the characteristic triplet determines the law of a Lévy processes, we obtain that a necessary and sufficient condition for (14) to hold is

$$\Pi(dy) = e^{-y}\Pi(-dy). \qquad (15)$$

This ensures $\tilde{\Pi} = \Pi$, and from this follows

$$a - (r - \delta) = \tilde{a} - (\delta - r),$$

giving (14), as we always have $\tilde{\sigma} = \sigma$. Condition (15) answers a question raised by Carr and Chesney (1996).

Examples and Applications

In this section we consider that the Lévy measure of the process has the form

$$\Pi(dy) = e^{\beta y}\Pi_0(dy),$$

where $\Pi_0(dy)$ is a symmetric measure, i.e.

$$\Pi_0(dy) = \Pi_0(-dy).$$

In many cases, the Lévy measure has a Radon-Nikodym density, and we have

$$\Pi(dy) = e^{\beta y}p(y)dy, \tag{16}$$

where $p(x) = p(-x)$, that is, the function $p(x)$ is even.

In this way, we want to model the asymmetry of the market through the parameter β. As a consequence of (15), we obtain that when $\beta = -1/2$ we have a symmetric market.

It is also interesting to note, that practically all parametric models proposed in the literature, in what concerns Lévy markets, including diffusions with jumps, can be reparametrized in the form (16) (with the exception of Kou (2000), see anyhow Kou and Wang (2001)).

Generalized Hyperbolic Model

This model has been proposed by Eberlein and Prause (1998) as they allow for a more realistic description of asset returns . This model has $\sigma = 0$, and a Lévy measure given by (16), with

$$p(y) = \frac{1}{|y|}\Big(\int_0^\infty \frac{\exp\big(-\sqrt{2z+\alpha^2}|y|\big)}{\pi^2 z\big(J^2_{|\lambda|}(\delta\sqrt{2z}) + Y^2_{|\lambda|}(\delta\sqrt{2z})\big)} dz + \mathbf{1}_{\{\lambda\geq 0\}}\lambda e^{-\alpha|y|}\Big),$$

Particular cases are the hyperbolic distribution, obtained when $\lambda = 1$; and the normal inverse gaussian when $\lambda = -1/2$.

The statistical estimation $\beta = -24.91$ is reported for the daily returns of the DAX (German stock index) for the period 15/12/93 to 26/11/97 (The other parameters are also estimated). This indicates the absence of symmetry.

The CGMY market model

This Lévy market model, proposed by Carr et al. (2002), is characterized by $\sigma = 0$ and Lévy measure given by (16), where the function $p(y)$ is given by

$$p(y) = \frac{C}{|y|^{1+Y}} e^{-\alpha|y|}.$$

The parameters satisfy $C > 0$, $Y < 2$, and $G = \alpha + \beta \geq 0$, $M = \alpha - \beta \geq 0$, where C, G, M, Y are the parameters used.

For studying the presence of a pure diffusion component in the model, condition $\sigma = 0$ is relaxed, and risk neutral distribution are estimated in a five parameters model. Values of $\beta = (G - M)/2$ are given for different assets, and in the general situation, the parameter β is negative, and less than $-1/2$.

Diffusions with jumps

Consider the jump–diffusion model proposed by Merton (1976). The driving Lévy process in this model has Lévy measure given by

$$\Pi(dy) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-(y-\mu)^2/(2\delta^2)},$$

and is direct to verify that condition (15) holds if and only if $2\mu + \delta^2 = 0$. This result was obtained by Bates (1997). The Lévy measure also corresponds to the form in (16), if we take $\beta = \mu/\delta^2$, and

$$p(y) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{\left(-(y^2+\mu^2)/(2\delta^2)\right)}.$$

A recent alternative jump distribution was proposed by Kou and Wang (2001). The Lévy measure has the form (16), where

$$p(y) = \lambda e^{-\alpha|y|}.$$

It can be observed that this is a particular case of the CGMY model, when $Y = -1$. In another model Eraker, Johansen and Polson (2000) introduce compound Poisson jumps into stochastic volatility processes, the Lévy measure is :

$$\Pi(dy) = \frac{\lambda}{\eta} e^{-\frac{y}{\eta}} dy, \quad y > 0$$

which is also a particular case of CGMY model.

Brazilian Data

- Fajardo and Farias (2002): *Generalized Hyperbolic Distributions and Brazilian Data*

Table 1: Estimated GHD Parameters.

Sample	α	β	δ	μ	λ	***LLH***
Bbas4	30.7740	3.5267	0.0295	-0.0051	-0.0492	3512.73
Bbdc4	47.5455	-0.0006	0	0	1	3984.49
Brdt4	56.4667	3.4417	0.0026	-0.0026	1.4012	3926.68
Cmig4	1.4142	0.7491	0.0515	-0.0004	-2.0600	3685.43
Csna3	46.1510	0.0094	0	0	0.6910	3987.52
Ebtp4	3.4315	3.4316	0.0670	-0.0071	-2.1773	1415.64
Elet6	1.4142	0.0120	0.0524	0	-1.8987	3539.06
Ibvsp	1.7102	-1.6684	0.0357	0.0020	-1.8280	4186.31
Itau4	49.9390	1.7495	0	0	1	4084.89
Petr4	7.0668	0.4848	0.0416	0.0003	-1.6241	3767.41
Tcsl4	1.4142	0	0.0861	0.0011	-2.6210	1329.64
Tlpp4	6.8768	0.4905	0.0359	0	-1.3333	3766.28
Tnep4	2.2126	2.2127	0.0786	-0.0028	-2.2980	1323.66
Tnlp4	1.4142	0.0021	0.0590	0.0005	-2.1536	1508.22
Vale5	25.2540	2.6134	0.0265	-0.0015	-0.6274	3958.47

- Fajardo, Schuschny and Silva (2001): *Lévy Processes and Brazilian Market*

Conclusions

- Geometric Lévy Motion
- Payoff function homogeneous of any degree
- Stochastic Bond
- Put-Call Duality
- Put-Call Symmetry

- Multidimensional Analysis
- Exotic Derivatives
- Dependent Increments