Skewness Premium with Lévy Processes

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Outline

1. Option Market
2. Lévy Processes
3. Duality
4. Symmetry
5. Skewness Premium
Observed moneyness biases in American call and put options: S&P500 options traded on CMEX
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American Foreign currency call options traded in Philadelphia Stock Exchange
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American Foreign currency call options traded in Philadelphia Stock Exchange

The Biases are not in the same direction, nor are they constant over time.
Some facts

- Out-of-the-money (OTM) Calls pays only if the asset price rises above the Call’s exercise price while OTM Puts pay off only if asset price falls below the Put’s exercise price.
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- Call and Put prices directly reflect characteristics of the upper and lower tails of the risk neutral distribution.
Some facts

- Out-of-the-money (OTM) Calls pays only if the asset price rises above the Call’s exercise price while OTM Puts pay off only if asset price falls below the Put’s exercise price.
- Call and Put prices directly reflects characteristics of the upper and lower tails of the risk neutral distribution.
- Then relative prices of OTM options will reflect the skewness of the risk neutral distribution.
Put-Call Parity:

\[ p + S = c + Xe^{-rT} \]
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Just for European Options!
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Put-Call Duality:

\[ c(S_0, K, r, \delta, \tau, \psi) = p(K, S_0, \delta, r, \tau, \tilde{\psi}) \]
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European and American Options!
Put-Call Parity:

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Just for European Options! **Same Strike**

Put-Call Duality:

\[ c(S_0, K, r, \delta, \tau, \psi) = p(K, S_0, \delta, r, \tau, \tilde{\psi}) \]

European and American Options! **Different Strike**
Call Options $x\%$ out-of-the-money are priced exactly $x\%$ higher than the corresponding OTM put:
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$$C(F, T; K_c) = (1 + x)P(F, T; K_p), \; x > 0$$
From Duality

Call Options $x\%$ out-of-the-money are priced exactly $x\%$ higher than the corresponding OTM put:

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Where $F$ is the future price:

$$K_c = F(1 + x) \text{ and } K_p = F/(1 + x). \text{ i.e. } K_cK_p = F^2.$$
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Bates’ $x\%$ rule!
Skewness Premium (SK): David S. Bates

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For which parameters $SK = \frac{C}{P} - 1 \leq 0$?
Option Market Data: 08/31/2006

Option Prices on S&P500, T=09/15/2006.
OTM options S&P500 - 08/31/06. T=09/15/06, F=1303.82

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<tr>
<th>$K_c$</th>
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OTM options S&P500 - 08/31/06. T= 09/15/06, F=1303.82

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**ITM options S&P500 - 08/31/06. T=09/15/06, F=1303.82**

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Skewness Premium (SK)

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- Asymmetry in the market.
Contribution

Theoretical proposition that quantify the relation between OTM Calls and Puts when the underlying follows a Geometric Lévy Process.
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Simply diagnostic for judging which distributions are consistent with observed option prices.
Consider a stochastic process \( X = \{X_t\}_{t \geq 0} \), defined on \((\Omega, \mathcal{F}, \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})\). We say that \( X = \{X_t\}_{t \geq 0} \) is a Lévy Process if:
Lévy Processes

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- $X$ has paths RCLL
- $X_0 = 0$, and has independent increments, given $0 < t_1 < t_2 < ... < t_n$, the r.v.

$$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

are independents.
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  \[ X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}} \]
  are independents.
- The distribution of the increment \( X_t - X_s \) is homogenous in time, that is, depends just on the difference \( t - s \).
Lévy-Khintchine Formula

A key result in the theory of Lévy Processes is the Lévy-Khintchine formula:

$$E(e^{zX_t}) = e^{t\psi(z)}$$
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Where $\psi$ is called *characteristic exponent*, and is given by:

$$\psi(z) = az + \frac{1}{2} \sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zy1_{\{|y|<1\}})\Pi(dy),$$
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Where \( \psi \) is called *characteristic exponent*, and is given by:

\[ \psi(z) = az + \frac{1}{2} \sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zy1_{\{|y|<1\}}) \Pi(dy), \]

where \( a \) and \( \sigma \geq 0 \) are real constants, and \( \Pi \) is a positive measure in \( \mathbb{R} - \{0\} \) such that \( \int (1 \wedge y^2) \Pi(dy) < \infty \), called the *Lévy measure*. 
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$$E(e^{zX_t}) = e^{t\psi(z)}$$

Where $\psi$ is called *characteristic exponent*, and is given by:

$$\psi(z) = a z + \frac{1}{2} \sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zy1_{\{|y|<1\}}) \Pi(dy),$$

where $a$ and $\sigma \geq 0$ are real constants, and $\Pi$ is a positive measure in $\mathbb{R} - \{0\}$ such that $\int (1 \wedge y^2) \Pi(dy) < \infty$, called the Lévy measure. The triplet $(a, \sigma^2, \Pi)$ is the characteristic triplet.
Consider a market with two assets given by

\[ S_t^1 = e^{X_t}, \text{ and } S_t^2 = S_0^2 e^{rt} \]

where \((X)\) is a one dimensional Lévy process, and for simplicity, and without loss of generality we take \(S_0^1 = 1\).
Consider a market with two assets given by

\[ S_t^1 = e^{X_t}, \quad \text{and} \quad S_t^2 = S_0^2 e^{rt} \]

where \((X)\) is a one dimensional Lévy process, and for simplicity, and without loss of generality we take \(S_0^1 = 1\).

In this model we assume that the stock pays dividends with constant rate \(\delta \geq 0\), and that the given probability measure \(\mathbb{Q}\) is the chosen equivalent martingale measure.
Duality

Denote by $\mathcal{M}_T$ the class of stopping times up to a fixed constant time $T$, i.e:

$$\mathcal{M}_T = \{\tau : 0 \leq \tau \leq T, \, \tau \text{ stopping time w.r.t } F\}$$

for the finite horizon case and for the perpetual case we take $T = \infty$ and denote by $\overline{\mathcal{M}}$ the resulting stopping times set.
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for the finite horizon case and for the perpetual case we take $T = \infty$ and denote by $\overline{\mathcal{M}}$ the resulting stopping times set. Then, for each stopping time $\tau \in \mathcal{M}_T$ we introduce

$$c(S_0, K, r, \delta, \tau, \psi) = E e^{-r\tau} (S_\tau - K)^+, \quad (1)$$

$$p(S_0, K, r, \delta, \tau, \psi) = E e^{-r\tau} (K - S_\tau)^+. \quad (2)$$
For the American finite case, prices and optimal stopping rules $\tau_c^*$ and $\tau_p^*$ are defined, respectively, by:

\[
C(S_0, K, r, \delta, T, \psi) = \sup_{\tau \in \mathcal{M}_T} E e^{-r\tau} (S_\tau - K)^+
= E e^{-r\tau_c^*} (S_{\tau_c^*} - K)^+ \quad (3)
\]

\[
P(S_0, K, r, \delta, T, \psi) = \sup_{\tau \in \mathcal{M}_T} E e^{-r\tau} (K - S_\tau)^+
= E e^{-r\tau_p^*} (K - S_{\tau_p^*})^+, \quad (4)
\]
And for the American perpetual case, prices and optimal stopping rules are determined by

\[
\overline{C}(S_0, K, r, \delta, \psi) = \sup_{\tau \in \mathcal{M}} \mathbb{E} e^{-r \tau} (S_{\tau} - K)^+ 1\{\tau < \infty\} \\
= \mathbb{E} e^{-r \tau^*_c} (S_{\tau^*_c} - K)^+ 1\{\tau < \infty\}, \tag{5}
\]

\[
\overline{P}(S_0, K, r, \delta, \psi) = \sup_{\tau \in \mathcal{M}} \mathbb{E} e^{-r \tau} (K - S_{\tau})^+ 1\{\tau < \infty\} \\
= \mathbb{E} e^{-r \tau^*_p} (K - S_{\tau^*_p})^+ 1\{\tau < \infty\}. \tag{6}
\]
Lemma (Duality)

Consider a Lévy market with driving process $X$ with characteristic exponent $\psi(z)$. Then, for the expectations introduced in (1) and (2) we have

$$c(S_0, K, r, \delta, \tau, \psi) = p(K, S_0, \delta, r, \tau, \tilde{\psi}),$$

where

$$\tilde{\psi}(z) = \tilde{a}z + \frac{1}{2}\tilde{\sigma}^2 z^2 + \int_{\mathbb{R}} \left(e^{zy} - 1 - zh(y))\tilde{\Pi}(dy\right)$$

is the characteristic exponent (of a certain Lévy process) that satisfies

$$\begin{cases}
\tilde{a} &= \delta - r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^y - 1 - h(y))\tilde{\Pi}(dy), \\
\tilde{\sigma} &= \sigma, \\
\tilde{\Pi}(dy) &= e^{-y}\Pi(-dy).
\end{cases}$$
Corollary (European Options)

For the expectations introduced in (1) and (2) we have

\[ c(S_0, K, r, \delta, T, \psi) = p(K, S_0, \delta, r, T, \tilde{\psi}) \],

(10)

with \( \psi \) and \( \tilde{\psi} \) as in the Duality Lemma.
Duality

**Corollary (European Options)**

For the expectations introduced in (1) and (2) we have

\[ c(S_0, K, r, \delta, T, \psi) = p(K, S_0, \delta, r, T, \tilde{\psi}), \]  

(10)

with \( \psi \) and \( \tilde{\psi} \) as in the Duality Lemma.

**Corollary (American Options)**

For the value functions in (3) and (4) we have

\[ C(S_0, K, r, \delta, T, \psi) = P(K, S_0, \delta, r, T, \tilde{\psi}), \]  

(11)

with \( \psi \) and \( \tilde{\psi} \) as in the Duality Lemma.
**Corollary (Perpetual Options)**

For prices of Perpetual Call and Put options in (5) and (6) the optimal stopping rules have, respectively, the form

\[
\tau_c^* = \inf \{ t \geq 0 : S_t \geq S_c^* \},
\]
\[
\tau_p^* = \inf \{ t \geq 0 : S_t \leq S_p^* \}.
\]

where the constants \( S_c^* \) and \( S_p^* \) are the critical prices. Then, we have

\[
\overline{C}(S_0, K, r, \delta, \psi) = \overline{P}(K, S_0, \delta, r, \tilde{\psi}), \quad (12)
\]

with \( \psi \) and \( \tilde{\psi} \) as in the Duality Lemma. Furthermore, when \( \delta > 0 \), for the optimal stopping levels, we obtain the relation \( S_c^* S_p^* = S_0 K \).
Dual markets

Given a Lévy market with driving process characterized by $\psi$, consider a market model with two assets, a deterministic savings account $\tilde{B} = \{\tilde{B}_t\}_{t \geq 0}$, given by

$$\tilde{B}_t = e^{\delta t}, \quad \delta \geq 0,$$

and a stock $\tilde{S} = \{\tilde{S}_t\}_{t \geq 0}$, modelled by

$$\tilde{S}_t = Ke^{\tilde{X}_t}, \quad \tilde{S}_0 = K > 0,$$
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Symmetric markets

Let's define symmetric markets by

\[ \mathcal{L}(e^{-(r-\delta)t+X_t} \mid \mathbb{Q}) = \mathcal{L}(e^{-(\delta-r)t-X_t} \mid \tilde{\mathbb{Q})}, \]  

meaning equality in law.
Symmetric markets

Lets define symmetric markets by

\[ \mathcal{L} \left( e^{-(r-\delta)t+X_t} \mid \mathcal{Q} \right) = \mathcal{L} \left( e^{-(\delta-r)t-X_t} \mid \bar{\mathcal{Q}} \right), \tag{13} \]

meaning equality in law.

A necessary and sufficient condition for (13) to hold is

\[ \Pi(dy) = e^{-y} \Pi(-dy), \tag{14} \]

This ensures \( \bar{\Pi} = \Pi \), and from this follows

\[ a - (r - \delta) = \bar{a} - (\delta - r) \]

, giving (13), as always \( \bar{\sigma} = \sigma \).
Bates’ x%-Rule

If the call and put options have strike prices x% out-of-the-money relative to the forward price, then the call should be priced x% higher than the put.
Bates’ $x\%$-Rule

If the call and put options have strike prices $x\%$ out-of-the-money relative to the forward price, then the call should be priced $x\%$ higher than the put.

If $r = \delta$, we can take the future price $F$ as the underlying asset in Lemma 1.
Bates’ x%-Rule

If the call and put options have strike prices x% out-of-the-money relative to the forward price, then the call should be priced x% higher than the put.

If \( r = \delta \), we can take the future price \( F \) as the underlying asset in Lemma 1.

Corollary

Take \( r = \delta \) and assume (14) holds, we have

\[
c(F, K_c, r, \tau, \psi) = (1 + x) \ p(F, K_p, r, \tau, \psi),
\]

(15)

where \( K_c = (1 + x)F \) and \( K_p = F/(1 + x) \), with \( x > 0 \).
Empirical Evidence of Symmetry

We restrict to Lévy markets with jump measure of the form

$$\Pi(dy) = e^{\beta y} \Pi_0(dy),$$

where $\Pi_0(dy)$ is a symmetric measure, i.e. $\Pi_0(dy) = \Pi_0(-dy)$, everything with respect to the risk neutral measure $\mathbb{Q}$. 

As a consequence of (14), market is symmetric if and only if $\beta = -1/2$. In view of this, we propose to measure the asymmetry in the market through the parameter $\beta + 1/2$. When $\beta + 1/2 = 0$ we have a symmetric market.
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In view of this, we propose to measure the asymmetry in the market through the parameter \( \beta + 1/2 \). When \( \beta + 1/2 = 0 \) we have a symmetric market.
We can obtain an Equivalent Martingale Measure by

\[ dQ_t = \frac{e^{\theta X_t}}{E^P e^{\theta X_t}} dP_t \]
Esscher Transform

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As a consequence:

\[ \beta_Q = \beta_P + \theta \]
Consider the jump - diffusion model proposed by Merton (1976). The driving Lévy process in this model has Lévy measure given by

\[ \Pi(dy) = \frac{1}{\delta \sqrt{2\pi}} e^{-(y-\mu)^2/(2\delta^2)} dy, \]

and is direct to verify that condition (14) holds if and only if \(2\mu + \delta^2 = 0\). This result was obtained by Bates (1997) for future options.
Consider the jump - diffusion model proposed by Merton (1976). The driving Lévy process in this model has Lévy measure given by

\[ \Pi(dy) = \lambda \frac{1}{\delta \sqrt{2\pi}} e^{-(y-\mu)^2/(2\delta^2)} dy, \]

and is direct to verify that condition (14) holds if and only if \( 2\mu + \delta^2 = 0 \). This result was obtained by Bates (1997) for future options.

That result is obtained as a particular case: \( \beta = \frac{\mu}{\delta^2} \)
Consider the Generalized Hyperbolic Distributions, with Lévy measure:

$$\Pi(dy) = e^{\beta y} \frac{1}{|y|} \left( \int_0^{\infty} \frac{\exp \left( -\sqrt{2z + \alpha^2 |y|} \right)}{\pi^2 z \left( J_\lambda^2 (\delta \sqrt{2z}) + Y_\lambda^2 (\delta \sqrt{2z}) \right)} dz + 1_{\{\lambda \geq 0\}} \lambda e^{-\alpha |y|} \right) dy$$

where $\alpha, \beta, \lambda, \delta$ are the historical parameters that satisfy the conditions $0 \leq |\beta| < \alpha$, and $\delta > 0$; and $J_\lambda$, $Y_\lambda$ are the Bessel functions of the first and second kind.
Example 1

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\]

where \(\alpha, \beta, \lambda, \delta\) are the historical parameters that satisfy the conditions \(0 \leq |\beta| < \alpha\), and \(\delta > 0\); and \(J_\lambda, Y_\lambda\) are the Bessel functions of the first and second kind.

Eberlein and Prause (1998): German Stocks
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Eberlein and Prause (1998): German Stocks
Fajardo and Farias (2004): Ibovespa

\[ \beta = -0.0035 \text{ and } \beta_Q = 80.65. \]
## Estimated Parameters GH

<table>
<thead>
<tr>
<th>Sample</th>
<th>(\alpha)</th>
<th>(\beta)</th>
<th>(\delta)</th>
<th>(\mu)</th>
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</table>
Example 2

Consider the Meixner distribution, with Lévy measure:

\[ \Pi(dy) = c \frac{e^{\frac{b}{a}y}}{y \sinh(\frac{\pi y}{a})} dy, \]

where \( a, b \) and \( c \) are parameters of the Meixner density, such that \( a > 0, -\pi < b < \pi \) and \( c > 0 \). Then \( \beta_p = b/a \).
Estimated Parameters Meixner

<table>
<thead>
<tr>
<th>Index</th>
<th>$\hat{a}$</th>
<th>$\hat{b}$</th>
<th>$\theta$</th>
<th>$\beta_Q + 1/2$</th>
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Example 3

This CGMY model, proposed by Carr et al. (2002) is characterized by $\sigma = 0$ and Lévy measure given by (1), where the function $p(y)$ is given by

$$p(y) = \frac{C}{|y|^{1+Y}} e^{-\alpha|y|}.$$  

The parameters satisfy $C > 0$, $Y < 2$, and $G = \alpha + \beta \geq 0$, $M = \alpha - \beta \geq 0$, where $C, G, M, Y$ are the parameters of the model.
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Values of $\beta = (G - M)/2$ are obtained for different assets under the market risk neutral measure and in the general situation, the parameter $\beta$ is negative and less than $-1/2$. 
Any model satisfying (14) must have identical Black-Scholes implicit volatilities for calls and puts with strikes $\ln(K_c/F) = \ln x = -\ln(K_p/F)$, with $x > 0$ arbitrary.
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That is, the volatility smile curve is symmetric in the moneyness $\ln(K/F)$.
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That is, the volatility smile curve is symmetric in the moneyness \( \ln(K/F) \).

By put-call parity, European calls and puts with same strike and maturity must have identical implicit volatilities.
The $x\%$ Skewness Premium is defined as the percentage deviation of $x\%$ OTM call prices from $x\%$ OTM put prices.
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\[
SK(x) = \frac{c(S, T; X_c)}{p(S, T; X_p)} - 1, \quad \text{for European Options,} \quad (16)
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Skewness Premium (SK)

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Skewness Premium (SK)

The SK was addressed for the following stochastic processes:

- Constant Elasticity of Variance (CEV), include arithmetic and geometric Brownian motion:

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- **Stochastic Volatility processes,** the benchmark model being those for which volatility evolves independently of the asset price:

  \[ dS_t = \mu S_t \, dt + \sigma_t S_t \, dB_t^1, \]
  \[ d\sigma_t = \delta(\theta - \sigma_t) \, dt + \vartheta \, dB_t^2 \]
Skewness Premium (SK)

- Jump-diffusion processes, the benchmark model is the Merton's (1976) model:

\[ dS_t = (\mu - \lambda \kappa) S_t dt + \sigma S_t dB_t + S_t dq_t \]

- \( \mu \) expected return on asset
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- \( \kappa \) mean size of jumps = \( E(Y - 1) \).
- \( \ln(Y) \) is Normal with variance \( \delta^2 \).
Some results

For European options in general and for American options on futures, the SK has the following properties for the above distributions.

- $SK(x) \leq x$ for CEV processes with $\rho \leq 1$. 
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1. $SK(x) \leq x$ for CEV processes with $\rho \leq 1$.
2. $SK(x) \leq x$ for jump-diffusions with log-normal jumps depending on whether $2\mu + \delta^2 \leq 0$. 
Some results

For European options in general and for American options on futures, the SK has the following properties for the above distributions.

- $SK(x) \leq x$ for CEV processes with $\rho \leq 1$.
- $SK(x) \leq x$ for jump-diffusions with log-normal jumps depending on whether $2\mu + \delta^2 \leq 0$.
- $SK(x) \leq x$ for Stochastic Volatility processes depending on whether $\rho_S \sigma \leq 0$. 
Some results

Now in equation (16) consider

\[ X_p = F(1 - x) < F < F(1 + x) = X_c, \quad x > 0. \]
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Then,

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\[ X_p = F(1-x) < F < F(1+x) = X_c, \quad x > 0. \]

Then,

- \( SK(x) < 0 \) for CEV processes only if \( \rho < 0 \).
- \( SK(x) \geq 0 \) for CEV processes only if \( \rho \geq 0 \).

When \( x \) is small, the two SK measures will be approx. equal. For in-the-money options (\( x < 0 \)), the propositions are reversed.

Calls \( x\% \) in-the-money should cost 0\% – \( x\% \) less than puts \( x\% \) in-the-money.
Proof

By duality we have:

\[ C(F, T; Fk, \sigma k^{1-\rho}, \rho, b) = kP(F, T; F/k, \sigma, \rho, b), \quad \text{for } k > 0. \]
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The idea is to exploit monotonicity of option prices on volatility. If \(k > 1\) and \(\rho < 1\) then \(\sigma k^{1-\rho} > \sigma\),

$$C(F, T; F_k, \sigma, \rho, b) < C(F, T; F_k, \sigma k^{1-\rho}, \rho, b)$$
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Main Problems

European option price is monotonically increasing in the volatility if and only if the option price, at each fixed time prior to maturity, is convex as function of the price of the underlying asset.

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Main results

Theorem

Take $r = \delta$ and assume $\Pi(dy) = \lambda F(dy)$, $\lambda > 0$, if $F$ is such that $\int e^y F(dy) \gtrsim 1$, then

$$c(F_0, K_c, r, \tau, \psi) \gtrsim (1 + x) p(F_0, K_p, r, \tau, \psi),$$  \hspace{1cm} (17)

where $K_c = (1 + x)F_0$ and $K_p = F_0/(1 + x)$, with $x > 0$. 
Proof

If \( \Pi(dy) = \lambda F(dy) \Rightarrow \tilde{\Pi}(dy) = e^{-y} \lambda F(-dy) \).
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If $\Pi(dy) = \lambda F(dy) \Rightarrow \tilde{\Pi}(dy) = e^{-y} \lambda F(-dy)$.

Let $\tilde{\lambda} = \lambda \int e^{-y} F(-dy)$ and $\tilde{F}(dy) = \frac{e^{-y} F(-dy)}{\int e^{-y} F(-dy)}$. Then,

$$\int e^{y} F(dy) \geq 1 \iff \tilde{\lambda} \geq \lambda.$$
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By Ekström and Tysk (2005), option prices are monotonic on jump intensity:
Proof

If $\Pi(dy) = \lambda F(dy)$ $\Rightarrow$ $\tilde{\Pi}(dy) = e^{-y} \lambda F(-dy)$.

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By Ekström and Tysk (2005), option prices are monotonic on jump intensity:

$$c(F_0, K_c, r, \tau, a, \sigma, \tilde{\lambda} \tilde{F}) \geq c(F_0, K_c, r, \tau, a, \sigma, \lambda F) = (1 + x)p(F_0, K_p, r, \tau, a, \sigma, \tilde{\lambda} \tilde{F})$$

where the last equality is obtained from duality.
Take $r = \delta$ and assume $\Pi(dy) = e^{\beta y} \Pi_0(dy)$, if $\beta \geq -1/2$, then

$$C(F, K_c, r, \tau, \psi) \geq (1 + x) P(F, K_p, r, \tau, \psi),$$  \hspace{1cm} (18)

where $K_c = (1 + x)F$ and $K_p = F/(1 + x)$, with $x > 0$. 

*Theorem*
Proof

We need monotonicity of call prices on the parameter $\beta$. 

We have $\beta \succ -\frac{1}{2} \iff \beta \succ \tilde{\beta} := -\beta - 1$, then,

$$\Pi(dy) = e^{\beta y} \Pi_0(dy)$$

has $\beta \succ \tilde{\beta}$ of $\tilde{\Pi} = e^{-\left(1+\beta\right)y} \Pi_0(dy)$. 

By monotonicity

$$c(F_0, K_c, r, \delta, \tau, a, \sigma, \Pi) \succ c(F_0, K_c, r, \tau, a, \sigma, \tilde{\Pi}) = (1+x)p(F_0, K_c, r, \tau, a, \sigma, \Pi),$$

where the last equality is obtained from duality and the fact that $\tilde{\tilde{\Pi}} = \Pi$. 
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We need monotonicity of call prices on the parameter $\beta$. We have

$$\beta \geq -1/2 \iff \beta \geq \tilde{\beta} := -\beta - 1,$$

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By monotonicity

$$c(F_0, K_c, r, \delta, \tau, a, \sigma, \Pi) \geq c(F_0, K_c, r, \tau, a, \sigma, \widetilde{\Pi}) = (1 + x) \rho(F_0, K_c, r, \tau, a, \sigma, \Pi),$$

where the last equality is obtained from duality and the fact that $\widetilde{\Pi} = \Pi$. 
Remember that \( \beta = \frac{\mu}{\delta^2} \), so we obtain that sufficient conditions in the above theorems are equivalent. That is,
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$$\int e^{-y} F(-dy) = e^{\mu + \delta^2/2} \geq 1 \iff \beta \geq -1/2.$$ 

In general it is not true.
Conclusions

- Symmetry and Bates’ x% Rule. ($\beta \neq -0.5$)
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Conclusions

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- The SK can not identify which process or which parameter values best fit observed option data.
Symmetry and Bates’ x% Rule. ($\beta \neq -0.5$)

Skewness Premium: Call option x% OTM should be priced $[0, x\%]$ more than Put options $x\%$ OTM. ($SK \not\geq 0$)

The SK can not identify which process or which parameter values best fit observed option data.

Which of the Lévy processes and associated option pricing models can generate the observed moneyness biases.
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- EMM Esscher Transform and Asymmetry.
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- EMM Esscher Transform and Asymmetry.
- Volatility Smile and Asymmetry
- Time-Changed Lévy Processes
- Other Derivatives
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Symmetries and pricing of exotic options in Lévy models.

E. Eberlein, A. Papapantoleon and A. N. Shiryaev
Universität Freiburg. WP 92. August 2006.
Let $Y$ be a $d$-dimensional semimartingale with finite variation with triplet $(B, C, \nu)$ under $P$, let $u, v$ be vectors in $\mathbb{R}^d$ and $v \in [-M, M]^d$. Moreover let $\tilde{P} \sim P$, with density $\frac{d\tilde{P}}{dP} = \frac{e^{\langle v, Y_T \rangle}}{E[e^{\langle v, Y_T \rangle}]}$. Then the process $Y^* := \langle u, Y \rangle$ is a $\tilde{P}$-semimartingale with triplet:

$$
\begin{align*}
    b_s^* &= \langle u, b_s \rangle + \frac{1}{2} (\langle u, c_s v \rangle + \langle v, c_s u \rangle) + \int_{\mathbb{R}^d} \langle u, x \rangle (e^{\langle v, x \rangle} - 1) \lambda_s(dx) \\
    c_s^* &= \langle u, c_s u \rangle \\
    \lambda_s^* &= \Lambda(\kappa_s)
\end{align*}
$$

where $\Lambda$ is a mapping $\lambda : \mathbb{R}^d \to \mathbb{R}$ such that $x \mapsto \Lambda(x) = \langle u, x \rangle$ and $\kappa_s$ is a measure defined by $\kappa_s(A) = \int_A e^{\langle v, x \rangle} \lambda_s(dx)$. 

**Lemma**