Skewness Premium with Lévy Processes

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Outline

- Motivation

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- Lévy processes

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- Duality and Symmetry

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- Conclusions

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- S&P500 options traded on CMEX
- American Foreign currency call options traded in Philadelphia Stock Exchange
- The Biases are not in the same direction, nor are they constant over time.

Some facts

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- Out-of-the-money (OTM) Calls pays only if the asset price rises above the Call's exercise price while OTM Puts pay off only if asset price falls below the Put's exercise price.
- Call and Put prices directly reflects characteristics of the upper and lower tails of the risk neutral distribution.
- Then relative prices of OTM options will reflect the skewness of the risk neutral distribution.

Put-Call relationship

Put-Call Parity:

$$p + S = c + Xe^{-rT}$$

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European and American Options!

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Just for European Options! Same Strike

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European and American Options! Different Strike

From Duality

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Where $K_c = F(1+x)$ and $K_p = F/(1+x)$.

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Where $K_c = F(1+x)$ and $K_p = F/(1+x)$.

Bates' $x\%$ rule!

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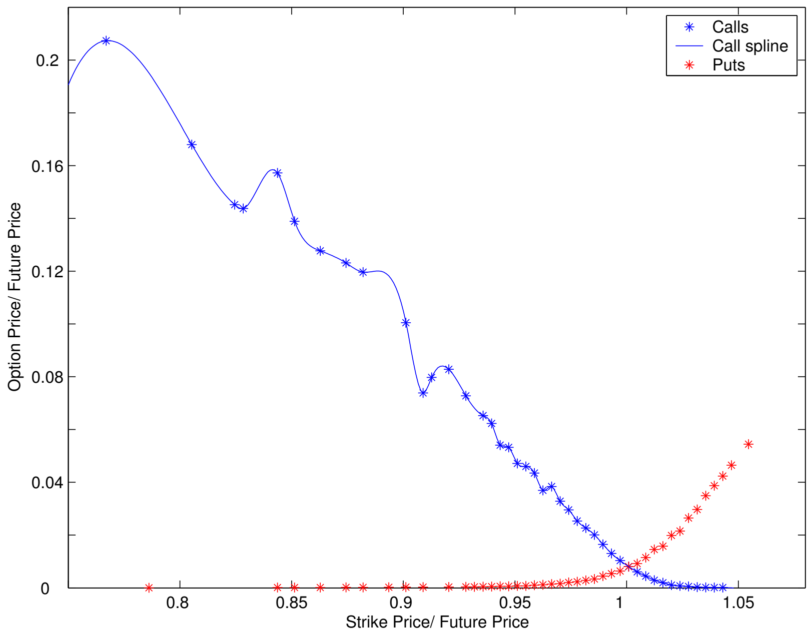
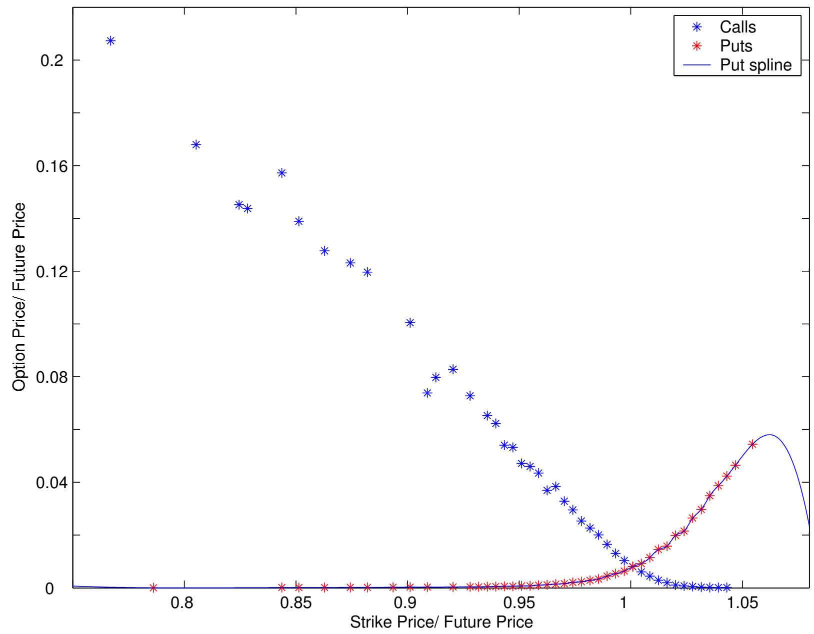
- For which parameters $SK = \frac{C}{P} - 1 \lessgtr 0$?

Interpolation





Option Prices on S&P500 in 08/31/2006.

Some facts: OTM options S&P500- Aug 31/06. T=Sept 15/06, F=1303.82

K_c	$K_p = F^2/K_c$	$x = K_c/F - 1$	$x_{obs} = c_{obs}/p_{int} - 1$	$x - x_{obs}$
1305	1302.641	0.000905	0.614561	-0.61366
1310	1297.669	0.00474	0.532798	-0.52806
1315	1292.735	0.008575	0.427299	-0.41872
1320	1287.838	0.01241	0.108911	-0.0965
1325	1282.979	0.016245	-0.11658	0.132826
1330	1278.155	0.020079	-0.45097	0.471053
1335	1273.368	0.023914	-0.50378	0.527697
1340	1268.617	0.027749	-0.61306	0.640807
1345	1263.901	0.031584	-0.73872	0.770305
1350	1259.22	0.035419	-0.81448	0.849896
1355	1254.573	0.039254	-0.80297	0.842224
1360	1249.961	0.043089	-0.82437	0.867454

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1250	1359.957	0.043056	-0.88837	0.931421
1255	1354.539	0.0389	-0.86897	0.907873
1260	1349.164	0.034778	-0.85655	0.891331
1265	1343.831	0.030688	-0.78107	0.81176
1270	1338.541	0.02663	-0.70531	0.731941
1275	1333.291	0.022604	-0.63926	0.661869
1280	1328.083	0.018609	-0.51726	0.535865
1285	1322.916	0.014646	-0.31216	0.326801
1290	1317.788	0.010713	-0.20329	0.214005
1295	1312.7	0.006811	-0.03659	0.043397
1300	1307.651	0.002938	0.090739	-0.0878

Some facts: ITM options S&P500- Aug 31/06. T=Sept 15/06, F=1303.82

K_c	$K_p = F^2/K_c$	$x = K_c/F - 1$	$x_{obs} = c_{obs}/p_{int} - 1$	$x - x_{obs}$
1230	1382.07	-0.05662	0.050681	-0.1073
1235	1376.475	-0.05278	0.13642	-0.1892
1240	1370.925	-0.04895	0.115006	-0.16395
1245	1365.419	-0.04511	0.197696	-0.24281
1250	1359.957	-0.04128	0.277944	-0.31922
1255	1354.539	-0.03744	0.280729	-0.31817
1260	1349.164	-0.03361	0.536286	-0.5699
1265	1343.831	-0.02977	0.574983	-0.60476
1270	1338.541	-0.02594	0.606719	-0.63266
1275	1333.291	-0.0221	0.675372	-0.69748
1280	1328.083	-0.01827	0.691325	-0.70959
1285	1322.916	-0.01443	0.966306	-0.98074
1290	1317.788	-0.0106	0.904839	-0.91544
1295	1312.7	-0.00676	0.794059	-0.80082
1300	1307.651	-0.00293	0.78018	-0.78311

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1305	1302.641	-0.0009	0.130843	-0.13175
1310	1297.669	-0.00472	0.252541	-0.25726
1315	1292.735	-0.0085	0.261905	-0.27041
1320	1287.838	-0.01226	0.242817	-0.25507
1325	1282.979	-0.01598	0.346419	-0.3624
1330	1278.155	-0.01968	0.183207	-0.20289
1335	1273.368	-0.02336	0.237999	-0.26135
1340	1268.617	-0.027	0.145858	-0.17286
1345	1263.901	-0.03062	0.152637	-0.18325
1350	1259.22	-0.03421	0.101211	-0.13542
1355	1254.573	-0.03777	-0.03964	0.001869
1360	1249.961	-0.04131	0.028337	-0.06965
1365	1245.382	-0.04482	-0.0101	-0.03472
1375	1236.325	-0.05177	-0.0451	-0.00667

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- ITM options: Usually, $x_{obs} > x$. That means $\frac{c}{p} - 1 > x$.
- Asset returns negatively skewed.

Contribution

- Theoretical proposition that quantify the relation between OTM Calls and Puts when the underlying follows a Geometric Lévy Process.

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- Theoretical proposition that quantify the relation between OTM Calls and Puts when the underlying follows a Geometric Lévy Process.
- Simply diagnostic for judging which distributions are consistent with observed option prices.

Lévy Processes

Consider a stochastic process $X = \{X_t\}_{t \geq 0}$, defined on $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$. We say that $X = \{X_t\}_{t \geq 0}$ is a Lévy Process if:

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- X has paths RCLL
- $X_0 = 0$, and has independent increments, given $0 < t_1 < t_2 < ... < t_n$, the r.v.

$$X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$$

are independents.

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  are independents.
- The distribution of the increment $X_t - X_s$ is homogenous in time, that is, depends just on the difference $t - s$.

Lévy-Khintchine Formula

A key result in the theory of Lévy Processes is the Lévy-Khintchine formula, that computes de characteristic function of X_t como:

$$E(e^{zX_t}) = e^{t\psi(z)}$$

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Where ψ is called *characteristic exponent*, and is given by:

$$\psi(z) = az + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zy\mathbf{1}_{\{|\mathbf{y}|<\mathbf{1}\}})\mathbf{\Pi}(\mathbf{dy}),$$

where b and $\sigma \geq 0$ are real constants, and Π is a positive measure in $\mathbb{R} - \{0\}$ such that $\int (1 \wedge y^2)\Pi(dy) < \infty$, called the *Lévy measure*. The triplet (a, σ^2, Π) is the *characteristic triplet.*

Model

Consider a market with two assets given by

$$S_t^1 = e^{X_t}, \text{ and } S_t^2 = S_0^2 e^{rt}$$

where (X) is a one dimensional Lévy process, and for simplicity, and without loss of generality we take $S_0^1 = 1$.

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In this model we assume that the stock pays dividends with constant rate $\delta \geq 0$, and that the given probability measure $\mathbb{Q}$ is the chosen equivalent martingale measure.

Duality

Denote by $\mathcal{M}_T$ the class of stopping times up to a fixed constant time T, i.e:

$$\mathcal{M}_T = \{\tau :\ 0 \leq \tau \leq T,\ \tau \text{ stopping time w.r.t } \mathbf{F}\}$$

for the finite horizon case and for the perpetual case we take $T = \infty$ and denote by $\overline{\mathcal{M}}$ the resulting stopping times set. Then, for each stopping time $\tau \in \mathcal{M}_T$ we introduce

$$c(S_0, K, r, \delta, \tau, \psi) = \mathbf{E}\, e^{-r\tau}(S_\tau - K)^+, \tag{1}$$

$$p(S_0, K, r, \delta, \tau, \psi) = \mathbf{E}\, e^{-r\tau}(K - S_\tau)^+. \tag{2}$$

Duality

For the American finite case, prices and optimal stopping rules τ_c^* and τ_p^* are defined, respectively, by:

$$
\begin{aligned}
C(S_0, K, r, \delta, T, \psi) &= \sup_{\tau \in \mathcal{M}_T} \mathbf{E}\, e^{-r\tau}(S_\tau - K)^+ \\
&= \mathbf{E}\, e^{-r\tau_c^*}(S_{\tau_c^*} - K)^+ \qquad (3) \\
P(S_0, K, r, \delta, T, \psi) &= \sup_{\tau \in \mathcal{M}_T} \mathbf{E}\, e^{-r\tau}(K - S_\tau)^+ \\
&= \mathbf{E}\, e^{-r\tau_p^*}(K - S_{\tau_p^*})^+, \qquad (4)
\end{aligned}
$$

Duality

And for the American perpetual case, prices and optimal stopping rules are determined by

$$\overline{C}(S_0, K, r, \delta, \psi) = \sup_{\tau \in \overline{\mathcal{M}}} \mathbf{E}\, e^{-r\tau}(S_\tau - K)^+ \mathbf{1}_{\{\tau<\infty\}}$$

$$= \mathbf{E}\, e^{-r\tau_c^*}(S_{\tau_c^*} - K)^+ \mathbf{1}_{\{\tau<\infty\}}, \quad (5)$$

$$\overline{P}(S_0, K, r, \delta, \psi) = \sup_{\tau \in \overline{\mathcal{M}}} \mathbf{E}\, e^{-r\tau}(K - S_\tau)^+ \mathbf{1}_{\{\tau<\infty\}}$$

$$= \mathbf{E}\, e^{-r\tau_p^*}(K - S_{\tau_p^*})^+ \mathbf{1}_{\{\tau<\infty\}}. \quad (6)$$

Put-Call Duality

Lemma 0.1 (Duality). *Consider a Lévy market with driving process X with characteristic exponent $\psi(z)$. Then, for the expectations introduced in* (1) *and* (2) *we have*

$$c(S_0, K, r, \delta, \tau, \psi) = p(K, S_0, \delta, r, \tau, \tilde{\psi}), \tag{7}$$

where

$$\tilde{\psi}(z) = \tilde{a}z + \frac{1}{2}\tilde{\sigma}^2 z^2 + \int \big(e^{zy} - 1 - zh(y)\big)\tilde{\Pi}(dy) \tag{8}$$

is the characteristic exponent (of a certain Lévy process) that satisfies

$$\begin{cases} \tilde{a} & = \delta - r - \sigma^2/2 - \int \big(e^y - 1 - h(y)\big)\tilde{\Pi}(dy), \\ \tilde{\sigma} & = \sigma, \\ \tilde{\Pi}(dy) & = e^{-y}\Pi(-dy). \end{cases} \tag{9}$$

Duality

Corollary 0.1 (European Options). *For the expectations introduced in* (1) *and* (2) *we have*

$$c(S_0, K, r, \delta, T, \psi) = p(K, S_0, \delta, r, T, \tilde{\psi}), \tag{10}$$

with ψ *and* $\tilde{\psi}$ *as in the Duality Lemma.*

Corollary 0.2 (American Options). *For the value functions in* (3) *and* (4) *we have*

$$C(S_0, K, r, \delta, T, \psi) = P(K, S_0, \delta, r, T, \tilde{\psi}), \tag{11}$$

with ψ *and* $\tilde{\psi}$ *as in the Duality Lemma.*

Duality

Corollary 0.3 (Perpetual Options). *For prices of Perpetual Call and Put options in* (5) *and* (6) *the optimal stopping rules have, respectively, the form*

$$\tau_c^* = \inf\{t \geq 0 \colon S_t \geq S_c^*\},$$
$$\tau_p^* = \inf\{t \geq 0 \colon S_t \leq S_p^*\}.$$

where the constants S_c^* *and* S_p^* *are the critical prices. Then, we have*

$$\overline{C}(S_0, K, r, \delta, \psi) = \overline{P}(K, S_0, \delta, r, \tilde{\psi}), \tag{12}$$

with ψ *and* $\tilde{\psi}$ *as in the Duality Lemma. Furthermore, when* $\delta > 0$*, for the optimal stopping levels, we obtain the relation*

$$S_c^* S_p^* = S_0 K. \tag{13}$$

Dual markets

Given a Lévy market with driving process characterized by ψ, consider a market model with two assets, a deterministic savings account $\tilde{B} = \{\tilde{B}_t\}_{t \geq 0}$, given by

$$\tilde{B}_t = e^{\delta t}, \qquad \delta \geq 0,$$

and a stock $\tilde{S} = \{\tilde{S}_t\}_{t \geq 0}$, modelled by

$$\tilde{S}_t = K e^{\tilde{X}_t}, \qquad \tilde{S}_0 = K > 0,$$

where $\tilde{X}_t = -X_t$ is a Lévy process with characteristic exponent under $\tilde{\mathbb{Q}}$ given by $\tilde{\psi}$ in (8). The process $\tilde{S}_t$ represents the price of KS_0 dollars measured in units of stock S.

Symmetric markets

Lets define symmetric markets by

$$\mathcal{L}\big(e^{-(r-\delta)t+X_t} \mid \mathbb{Q}\big) = \mathcal{L}\big(e^{-(\delta-r)t-X_t} \mid \tilde{\mathbb{Q}}\big), \tag{14}$$

meaning equality in law.

A necessary and sufficient condition for (14) to hold is

$$\Pi(dy) = e^{-y}\Pi(-dy), \tag{15}$$

This ensures $\tilde{\Pi} = \Pi$, and from this follows

$$a - (r - \delta) = \tilde{a} - (\delta - r)$$

, giving (14), as always $\tilde{\sigma} = \sigma$.

Bates' $x\%$-Rule

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If $r = \delta$, we can take the future price F as the underlying asset in Lemma 1.

Corollary 0.6. *Take $r = \delta$ and assume (15) holds, we have*

$$\mathcal{C}(F_0, K_c, r, \tau, \psi) = x\,\mathcal{P}(F_0, K_p, r, \tau, \psi), \tag{18}$$

where $K_c = xF_0$ and $K_p = F_0/x$, with $x > 0$.

Diffusions with jumps

Consider the jump - diffusion model proposed by Merton (1976). The driving Lévy process in this model has Lévy measure given by

$$\Pi(dy) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-(y-\mu)^2/(2\delta^2)} dy,$$

and is direct to verify that condition (15) holds if and only if $2\mu + \delta^2 = 0$. This result was obtained by Bates (1997) for future options.

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That result is obtained as a particular case, if we re place the future price as being the underlying asset, when $r = \delta$ in Lemma 1.

Lévy Processes

We restrict to Lévy markets with jump measure of the form

$$\Pi(dy) = e^{\beta y}\Pi_0(dy),$$

where $\Pi_0(dy)$ is a symmetric measure, i.e. $\Pi_0(dy) = \Pi_0(-dy)$, everything with respect to the risk neutral measure $\mathbb{Q}$.

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In view of this, we propose to measure the *asymmetry* in the market through the parameter $\beta + 1/2$. When $\beta + 1/2 = 0$ we have a symmetric market.

Esscher Transform

We can obtain an Equivalent Martingale Measure by

$$d\mathbb{Q}_t = \frac{e^{\theta X_t}}{\mathbf{E}^{\mathbb{P}}\, e^{\theta X_t}} d\mathbb{P}_t$$

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There is a θ such that the discounted price process is a martingale respect to $\mathbb{Q}$.

As a consequence:

$$\beta_{\mathbb{Q}} = \beta_{\mathbb{P}} + \theta$$

Example 1

Consider the Generalized Hyperbolic Distributions, with Lévy measure:

$$\Pi(dy) = e^{\beta y}\frac{1}{|y|}\Big(\int_0^\infty \frac{\exp\big(-\sqrt{2z+\alpha^2}|y|\big)}{\pi^2 z\big(J_\lambda^2(\delta\sqrt{2z}) + Y_\lambda^2(\delta\sqrt{2z})\big)}dz + \mathbf{1}_{\{\lambda\geq 0\}}\lambda e^{-\alpha|y|}\Big)dy$$

where $\alpha, \beta_{\mathbb{P}}, \lambda, \delta$ are the historical parameters that satisfy the conditions $0 \leq |\beta_{\mathbb{P}}| < \alpha$, and $\delta > 0$; and J_λ, Y_λ are the Bessel functions of the first and second kind.

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Eberlein and Prause (1998): German Stocks

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Consider the Generalized Hyperbolic Distributions, with Lévy measure:

$$\Pi(dy) = e^{\beta y}\frac{1}{|y|}\Big(\int_0^\infty \frac{\exp\big(-\sqrt{2z+\alpha^2}|y|\big)}{\pi^2 z\big(J_\lambda^2(\delta\sqrt{2z}) + Y_\lambda^2(\delta\sqrt{2z})\big)}dz + \mathbf{1}_{\{\lambda\geq 0\}}\lambda e^{-\alpha|y|}\Big)dy$$

where $\alpha, \beta_{\mathbb{P}}, \lambda, \delta$ are the historical parameters that satisfy the conditions $0 \leq |\beta_{\mathbb{P}}| < \alpha$, and $\delta > 0$; and J_λ, Y_λ are the Bessel functions of the first and second kind.

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$$\beta_{\mathbb{P}} = -0.0035 \text{ and } \beta_{\mathbb{Q}} = 80.65.$$

Parametros Estimados GH

Sample	α	β	δ	μ	λ	**LLH**
Bbas4	30.7740	3.5267	0.0295	-0.0051	-0.0492	3512.73
Bbdc4	47.5455	-0.0006	0	0	1	3984.49
Brdt4	56.4667	3.4417	0.0026	-0.0026	1.4012	3926.68
Cmig4	1.4142	0.7491	0.0515	-0.0004	-2.0600	3685.43
Csna3	46.1510	0.0094	0	0	0.6910	3987.52
Ebtp4	3.4315	3.4316	0.0670	-0.0071	-2.1773	1415.64
Elet6	1.4142	0.0120	0.0524	0	-1.8987	3539.06
Ibvsp	1.7102	-0.0035	0.0357	0.0020	-1.8280	4186.31
Itau4	49.9390	1.7495	0	0	1	4084.89
Petr4	7.0668	0.4848	0.0416	0.0003	-1.6241	3767.41
Tcsl4	1.4142	0	0.0861	0.0011	-2.6210	1329.64
Tlpp4	6.8768	0.4905	0.0359	0	-1.3333	3766.28
Tnep4	2.2126	2.2127	0.0786	-0.0028	-2.2980	1323.66
Tnlp4	1.4142	0.0021	0.0590	0.0005	-2.1536	1508.22
Vale5	25.2540	2.6134	0.0265	-0.0015	-0.6274	3958.47

Example 2

Consider the Meixner distribution, with Lévy measure:

$$\Pi(dy) = c\frac{e^{\frac{b}{a}y}}{y\sinh(\pi y/a)}dy,$$

where a, b and c are parameters of the Meixner density, such that $a > 0$, $-\pi < b < \pi$ and $c > 0$. Then $\beta_{\mathbb{P}} = b/a$.

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Index	$\hat{a}$	$\hat{b}$	θ	$\beta_{\mathbb{Q}} + 1/2$
Nikkei 225	0.02982825	0.12716244	0.42190524	5.18506
DAX	0.02612297	-0.50801886	-4.46513538	-23.4123
FTSE-100	0.01502403	-0.014336370	-4.34746821	-4.8017
Nasdaq Comp.	0.03346698	-0.49356259	-5.95888693	-20.2066
CAC-40.	0.02539854	-0.23804755	-5.77928595	-14.6518

Schoutens (2002) estimates with data $1/1/1997$ to $12/31/1999$

Example 3

This CGMY model, proposed by Carr et al. (2002) is characterized by $\sigma = 0$ and Lévy measure given by (28), where the function $p(y)$ is given by

$$p(y) = \frac{C}{|y|^{1+Y}} e^{-\alpha|y|}.$$

The parameters satisfy $C > 0$, $Y < 2$, and $G = \alpha + \beta \geq 0$, $M = \alpha - \beta \geq 0$, where C, G, M, Y are the parameters of the model.

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Values of $\beta = (G - M)/2$ are obtained for different assets under the market risk neutral measure and in the general situation, the parameter β is negative and less than $-1/2$.

Implied volatility

- Any model satisfying (15) must have identical Black-Scholes implicit volatilities for calls and puts with strikes $\ln(K_c/F) = \ln x = -\ln(K_p/F)$, with $x > 0$ arbitrary.

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- That is, the volatility smile curve is symmetric in the moneyness $\ln(K/F)$.
- By put-call parity, European calls and puts with same strike and maturity must have identical implicit volatilities.

Skewness Premium (SK)

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$$SK(x) = \frac{c(S,T;X_c)}{p(S,T;X_p)} - 1, \quad \text{for European Options,} \qquad (20)$$

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where $X_p = \frac{F}{(1+x)} < F < F(1+x) = X_c, \quad x > 0$

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- Jump-diffusion processes, the benchmark model is the Merton's (1976) model.

Some results

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Calls $x\%$ in-the-money should cost $0\% - x\%$ less than puts $x\%$ in-the-money.

Some results

Theorem 0.1. *Take* $r = \delta$ *and assume that in the particular case (28), If* $\beta \gtreqless -1/2$*, then*

$$c(F_0, K_c, r, \tau, \psi) \gtreqless (1 + x)\; p(F_0, K_p, r, \tau, \psi), \tag{22}$$

where $K_c = (1 + x)F_0$ *and* $K_p = F_0/(1 + x)$*, with* $x > 0$*.*

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- Time-Changed Lévy Processes
- Other Derivatives

References

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References

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References

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References

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- Eberlein, Papapantoleon and Shiryaev (2006). *On the Duality Principle in Option Pricing: Semimartingale Setting*. Universität Freiburg. WP 92.