The Mean-Variance Frontier
Asset Pricing
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Lets verify that the relationship between the expected return of a portafolio μ_λ and its standard deviation σ_λ is a quadratic relationship:

Let λ and $1-\lambda$ the proportions invested in asset A and B, which have expected returns denoted by μ_a e μ_b, respectivelly. Then

$$\mu_\lambda = \lambda\mu_a + (1-\lambda_b)\mu_b = \mu_b + \lambda(\mu_b - \mu_a).$$

Then:

$$\lambda = \frac{\mu_\lambda - \mu_a}{\mu_b - \mu_a} \quad \text{and} \quad 1-\lambda = \frac{\mu_b - \mu_\lambda}{\mu_b - \mu_a}.$$

Using the formula of a variance of a portfolio with correlation ρ_{ab}, we obtain;

$$\sigma_\lambda^2 = \lambda^2\sigma_a^2 + (1-\lambda)^2\sigma_b^2 + 2\lambda(1-\lambda)\sigma_a\sigma_b\rho_{ab}$$

After math gives:

$$\sigma_c^2 = \alpha\mu_\lambda^2 + \beta\mu_\lambda + \gamma,$$

where

$$\begin{aligned}
\alpha &= \frac{\sigma_a^2 + \sigma_b^2 - 2\sigma_a\sigma_b\rho_{ab}}{(\mu_b - \mu_a)^2} > 0 \\
\beta &= \frac{-2[\sigma_a^2\mu_b + \sigma_b^2\mu_a - (\mu_b + \mu_a)\sigma_a\sigma_b\rho_{ab}]}{(\mu_b - \mu_a)^2} \\
\gamma &= \frac{\sigma_a^2\mu_b^2 + \sigma_b^2\mu_a^2 - 2\mu_b\mu_a\sigma_a\sigma_b\rho_{ab}}{(\mu_b - \mu_a)^2} > 0.
\end{aligned}$$

From here:

- As $\sigma_\lambda^2 \geq 0$, can not be more than one root.
- Deriving the variance in relation to μ_λ and equalizing to zero, we have the minimum variance when:

$$\mu_\lambda = -\frac{\beta}{2\alpha}$$

- And the portfolio of minimum variance has proportion:

$$\lambda = -\frac{1}{2}\frac{\beta/\alpha + 2\mu_a}{\mu_b - \mu_a}$$

- The resulting minimum variance will be:

$$\sigma^2_{min} = \gamma - \frac{\beta^2}{4\alpha}$$