Implied Volatility with Jumps

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Financial Returns: Stylized Fact

- Asymmetric distribution
- Excess of Kurtosis
- Presence of jumps
- Volatility clustering

Lévy Processes

Consider a stochastic process $X = \{X_t\}_{t \geq 0}$, defined on $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbf{Q})$. We say that $X = \{X_t\}_{t \geq 0}$ is a Lévy Process if:

- X has paths RCLL
- $X_0 = 0$, and has independent increments, given $0 < t_1 < t_2 < ... < t_n$, the r.v.

$$X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$$

  are independents.
- The distribution of the increment $X_t - X_s$ is homogenous in time, that is, depends just on the difference $t - s$.

Lévy-Khintchine Formula

A key result in the theory of Lévy Processes is the Lévy-Khintchine formula:

$$E(e^{zX_t}) = e^{t\psi(z)}$$

Where ψ is called *characteristic exponent*, and is given by:

$$\psi(z) = az + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zy\mathbf{1}_{\{|y|<1\}})\Pi(dy),$$

where a and $\sigma \geq 0$ are real constants, and Π is a positive measure in $\mathbb{R} - \{0\}$ such that $\int (1 \wedge y^2)\Pi(dy) < \infty$, called the *Lévy measure*. The triplet (a, σ^2, Π) is the *characteristic triplet.*

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Poisson Jumps

Consider the jump - diffusion model proposed by Merton (1976). The driving Lévy process in this model has Lévy measure given by

$$\Pi(dy) = \lambda \frac{1}{\delta\sqrt{2\pi}} e^{-(y-\mu)^2/(2\delta^2)} dy,$$

Since, in this model the Lévy measure belongs to our class, with $\beta = \frac{\mu}{\delta^2}$. Then, applying our Theorem the result follows.

Infinite Activity Jumps

Consider the Generalized Hyperbolic Distributions, with Lévy measure:

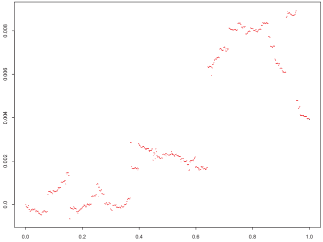
$$\Pi(dy) = e^{\beta y} \frac{1}{|y|} \Big(\int_0^{\infty} \frac{\exp\big(-\sqrt{2z+\alpha^2}|y|\big)}{\pi^2 z (J_\lambda^2(\delta\sqrt{2z}) + Y_\lambda^2(\delta\sqrt{2z}))} dz + \mathbf{1}_{\{\lambda \geq 0\}} \lambda e^{-\alpha|y|} \Big) dy$$

where $\alpha, \beta_{\mathbf{P}}, \lambda, \delta$ are the historical parameters that satisfy the conditions $0 \leq |\beta_{\mathbf{P}}| < \alpha$, and $\delta > 0$; and J_λ, Y_λ are the Bessel functions of the first and second kind.

Eberlein and Prause (1998): German Stocks
Fajardo and Farias (2004): Ibovespa

Generalized hyperbolic Lévy motion

The (Lévy) process generated by the generalized hyperbolic distribution, is a pure jump Lévy process with Lévy triplet $(\mathbb{E}[GH], 0, \nu^{GH})$.



Simulated path of a generalized hyperbolic Lévy motion.

The paths have infinitely many *small* jumps on every $[0, T]$ and infinite variation.

Lévy Market Models

- Lévy process to model risky assets
- Mathematically tractable
- Reasonable computational effort
- More realistic asset price model

Model

Consider a market with two assets given by

$$S_t^1 = e^{X_t}, \text{ and } S_t^2 = S_0^2 e^{rt}$$

where (X) is a one dimensional Lévy process, and for simplicity, and without loss of generality we take $S_0^1 = 1$.

In this model we assume that the stock pays dividends with constant rate $\delta \geq 0$, and that the given probability measure $\mathbf{Q}$ is the chosen equivalent martingale measure.

Dual markets

Given a Lévy market with driving process characterized by ψ, consider a market model with two assets, a deterministic savings account $\tilde{B} = \{\tilde{B}_t\}_{t\geq 0}$, given by

$$\tilde{B}_t = e^{\delta t}, \qquad \delta \geq 0,$$

and a stock $\tilde{S} = \{\tilde{S}_t\}_{t\geq 0}$, modelled by

$$\tilde{S}_t = Ke^{\tilde{X}_t}, \qquad \tilde{S}_0 = K > 0,$$

where $\tilde{X}_t = -X_t$ is a Lévy process with characteristic exponent under $\tilde{\mathbf{Q}}$ given by $\tilde{\psi}$.The process $\tilde{S}_t$ represents the price of KS_0 dollars measured in units of stock S.

Symmetric markets: FM (2006)

Lets define symmetric markets by

$$\mathcal{L}\left(e^{-(r-\delta)t+X_t} \mid \mathbf{Q}\right) = \mathcal{L}\left(e^{-(\delta-r)t-X_t} \mid \tilde{\mathbf{Q}}\right), \tag{1}$$

meaning equality in law.

When the underlying is a Lévy process, a necessary and sufficient condition for (1) to hold is

$$\Pi(dy) = e^{-y}\Pi(-dy), \tag{2}$$

This ensures $\tilde{\Pi} = \Pi$, and from this follows

$$a - (r - \delta) = \tilde{a} - (\delta - r)$$

, giving (1), as always $\tilde{\sigma} = \sigma$.

Symmetric Implied Volatility

Equivalence between market symmetry, put-call symmetry and symmetric implied volatility w.r.t log-moneyness.

- Fajardo and Mordecki (2006): Lévy Processes.
- Carr and Lee (2009): Local/Stochastic volatility models and Time Changed Lévy processes.
- Molchanov and Schmutz (2010): Multivariate Extension

Also under Symmetry

- Static Hedging for Exotic options: Carr and Bowie (1995), Carr, Ellis and Gupta (1998), Schmutz (2011).
- Robust replication for Vol derivatives: Carr and Lee (2009).
- Skewness Premium: Bates (1997), Fajardo and Mordecki (2012).

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Empirical Evidence of Market Symmetry: FM (2010)

Lévy markets with jump measure of the form

$$\Pi(dy) = e^{\beta y}\Pi_0(dy), \tag{3}$$

where $\Pi_0(dy)$ is a symmetric measure, i.e. $\Pi_0(dy) = \Pi_0(-dy)$, everything with respect to the risk neutral measure **Q**.

As a consequence of (2), market is symmetric if and only if $\beta = -1/2$.

Implied Volatility Smirk

The quadratic implied volatility approximation presented by Foresi and Wu (2005) is given by

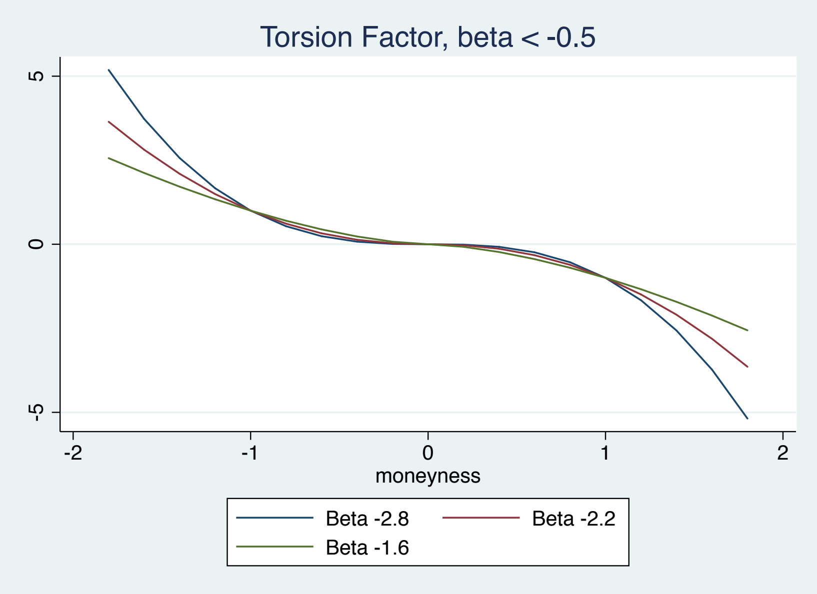
$$\sigma_{imp}(x_i, -0.5) = \gamma_0 + \gamma_1 d_i + \gamma_2 d_i^2 + e_i, \tag{4}$$

where $d_i = \frac{x_i}{\bar{\sigma}\sqrt{T}}$, is the standarized moneyness, $\bar{\sigma}$ is an average volatility and e_i is a normal distributed error. They called γ_0, γ_1 and γ_2, level, slope and curvature, respectively.

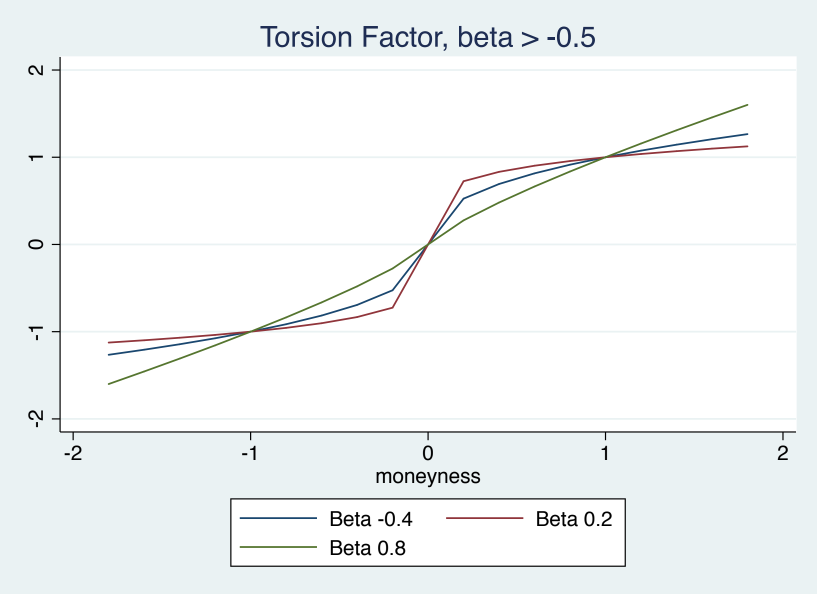
Fajardo (2017),“ A New Factor to Explain Implied Volatility Smirk”:,
consider the model

$$\sigma_{imp}(x_i, \beta) = \gamma_0 + \gamma_1 d_i + \gamma_2 d_i^2 + \gamma_3 (d_i + 1)^{\beta+0.5} + e_i,$$

Torsion Factor



Torsion Factor



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Numerical Example: NIG

Characteristic function:

$$\psi(z) = iz\mu + \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+iz)^2}\right), \ \beta-\alpha < \mathrm{Im}(z) < \beta+\alpha. \qquad (5)$$

Recall that the definition of NIG distribution requires $\delta > 0$ $\alpha > 0$ and $|\beta| < \alpha$. We can verify that $2\beta i$ belongs to the strip $\{z \in \mathbb{C} : \beta-\alpha < \mathrm{Im}(z) < \beta+\alpha\}$ and $\psi(2\beta i)$ is well defined and finite.
And from the martingale condition we have:

$$\psi(z) = \delta\left(iz\sqrt{\alpha^2-(\beta+1)^2} + (1-iz)\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+iz)^2}\right).$$

We consider daily returns and implied volatilities of S&P500 extracted from Bloomberg. We pick randomly the sample period 12/01/2009 to 12/01/2011 and using maximum likelihood estimation we find:

$$(\mu, \alpha, \delta, \beta) = (0.0018, 49.99, 0.0085, -9.22).$$

But we need the risk-neutral parameters, so we use the density given by the Esscher Transform.

$$(\mu^*, \alpha^*, \delta^*, \beta^*) = (0.0018, 49.99, 0.0085, -4.18). \qquad (6)$$

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Option and Implied Volatility Data

For the Taylor approximation we use the implied volatilities from the ATM contracts of S&P500 and maturities from 15 to 386 days. The implied volatilities observed at 12/01/2011:

Table: Option Data

Maturity	T	F	σ_{imp}^{ATM}	r	δ
12/16/2011	0.0411	1242.87	24.67%	0.208%	1.619%
01/20/2012	0.1370	1242.06	24.52%	0.344%	1.826%
02/17/2012	0.2137	1239.90	25.58%	0.454%	2.218%
03/16/2012	0.2904	1237.97	26.16%	0.498%	2.328%
06/15/2012	0.5397	1232.28	26.64%	0.540%	2.367%
09/21/2012	0.8082	1226.62	26.52%	0.580%	2.358%
12/21/2012	1.0575	1221.18	26.45%	0.604%	2.370%

Source: Bloomberg

Option and Implied Volatility Data

- We use liquid options on SP500 from Bloomberg quoted on 12/01/2011.
- The lowest strike is selected from the first out-of-the-money put with non-zero bid price. The highest price is selected from the first out-of-the-money call with non-zero bid price.
- Also, the call and put prices are the mid-value of closing bid and ask prices.
- We use the closing VIX of 12/01/2011, as a proxy for the average volatility, $\bar{\sigma} = 27.41\%$.

Option Data

Table: Options for Quadratic Approximation

	K^{lowest}	$K^{highest}$	N
12/16/2011	880	1430	40
01/20/2012	845	1500	46
02/17/2012	500	1600	36
03/16/2012	500	1500	35

Source: Bloomberg

Option and Implied Volatility Data

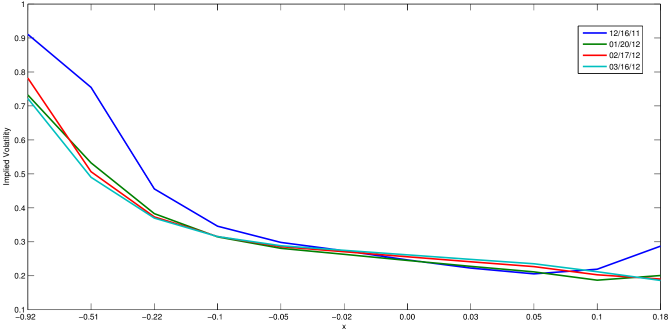


Figure: Impled Volatility Term Structure

Implied Volatility Quadratic Approximation

Table: Estimated Parameters by Maturity

	(1) mat15	(2) mat50	(3) mat78	(4) mat106
Level	0.2291***	0.2195***	0.2388***	0.2433***
	(0.0009)	(0.0007)	(0.0003)	(0.0003)
Slope	-0.0400***	-0.0494***	-0.0609***	-0.0713***
	(0.0016)	(0.0016)	(0.0014)	(0.0014)
Curvature	0.0135***	0.0142***	0.0095***	0.0119***
	(0.0012)	(0.0012)	(0.0015)	(0.0019)
N	40	46	36	35
R^2-adj	0.9396	0.9631	0.9950	0.9948

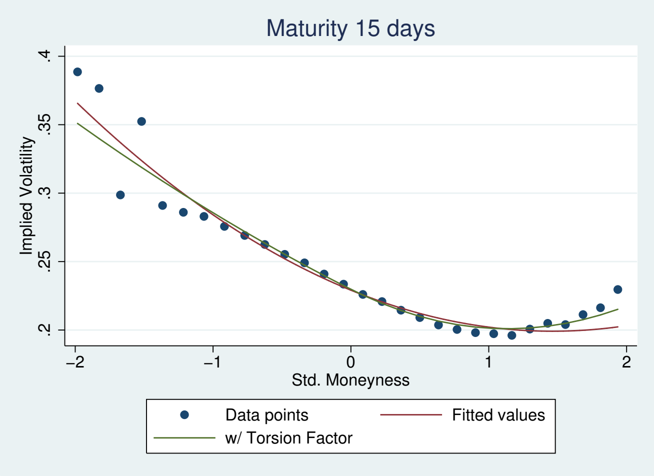
Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

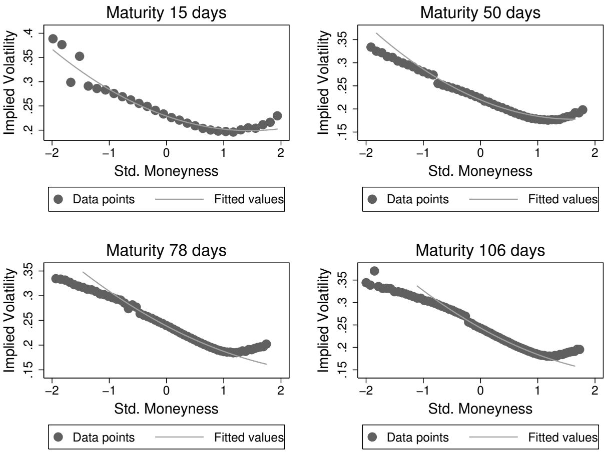
Implied Volatility with Torsion

	(1) mat16	(2) mat51	(3) mat79	(4) mat107	(5) mat19
Level	0.2296***	0.2203***	0.2394***	0.2437***	0.2455*
	(0.0006)	(0.0007)	(0.0005)	(0.0004)	(0.000
Slope	-0.0504***	-0.0556***	-0.0639***	-0.0737***	-0.0647
	(0.0019)	(0.0025)	(0.0020)	(0.0017)	(0.001
Curvature	0.0220***	0.0188***	0.0120***	0.0149***	-0.002
	(0.0015)	(0.0018)	(0.0019)	(0.0021)	(0.001
Torsion	-0.0000**	-0.0000	-0.0004	-0.0001	0.0003
	(0.0000)	(0.0000)	(0.0002)	(0.0001)	(0.000
N	24	33	34	34	19
r2_a	0.9771	0.9668	0.9955	0.9959	0.999
F	328.7946	311.1946	2414.5261	2654.9499	23361.7
ll	112.0887	144.8929	174.3269	188.9655	133.26

Implied Volatility Quadratic Approximation



Implied Volatility Quadratic Approximation



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Implied Volatility Quadratic Approximation

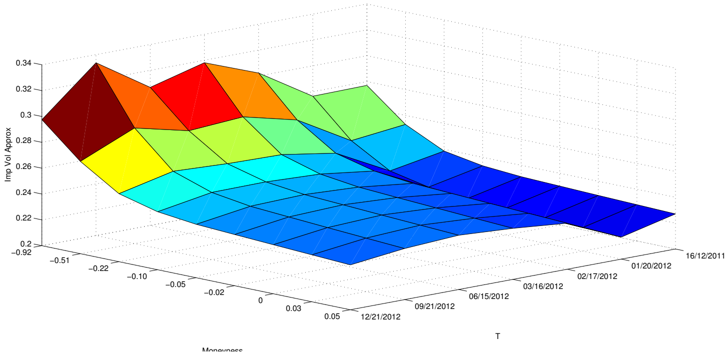

Figure: Impled Volatility Term Structure