Two-Period Model:
Mean-Variance Approach

千里之行始于足下

"A journey of a thousand miles starts with the first step."
Chinese proverb

Overview on assumptions

- Two-period model:
  - First we invest into assets
  - Then the assets pay off
- Mean-variance preferences:
  - done in practice
  - some shortcomings

CAPM

- Foundation: Markowitz mean-variance analysis (1952)
- Markowitz recommends the use of an expected return-variance of return rule,

  > *. . . both as a hypothesis to explain well-established investment behavior and as a maxim to guide one's own action.*

- Nobel prize: 1990 to Markowitz and Sharpe
- Main point: excess returns are explained by covariance to market portfolio.

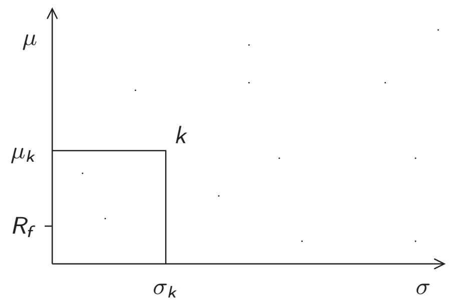
Mathematics of the CAPM

We start with an intuitive approach before we discuss more formal derivations:

$k = 1, 2, \ldots, K$	assets
$R_k := A_k / q_k$	gross return of asset k
q_k	first period market price
A_k	second period payoff
$\mu_k := \mu(R_k)$	expected return
$\sigma_k^2 := \mathrm{var}\,(R_k)$	variance

Geometric Intuition for the CAPM

Assets can be represented in a two-dimensional diagram.



Attractiveness of a single asset is characterized by mean and standard deviation.

Risk free-asset has an expected return of R_f with a zero standard deviation.

Diversification – History

There was a time when diversification as a means of reducing risk was not universally accepted. Markowitz' portfolio theory and their risk diversification was very controversial.

> *To suppose that safety-first consists of having a small gamble in a large number of different [companies] . . . strikes me as a travesty of investment policy.*
> *— J. M. Keynes [Keynes, 1988].*

Later the impact of the idea of diversification made such criticism look queer.

Diversification – Introduction

If we combine two risky assets k and j we obtain an expected portfolio return of $\mu_\lambda := \lambda\mu_k + (1-\lambda)\mu_j$, where λ is the portion of wealth invested in asset k. The portfolio variance is

$$\sigma_\lambda^2 := \lambda^2\sigma_k^2 + (1-\lambda)^2\sigma_j^2 + 2\lambda(1-\lambda)\,\mathrm{cov}_{k,j}\,.$$

How much one can gain by combining risky assets depends on covariance:

smaller covariance
$\rightsquigarrow$ higher *diversification* potential of mixing risky assets.

Correlation

It is convenient to standardize the covariance with the standard deviation.
The *correlation*

$$\mathrm{corr}_{k,j} := \mathrm{cov}_{k,j} / (\sigma_k \sigma_j)$$

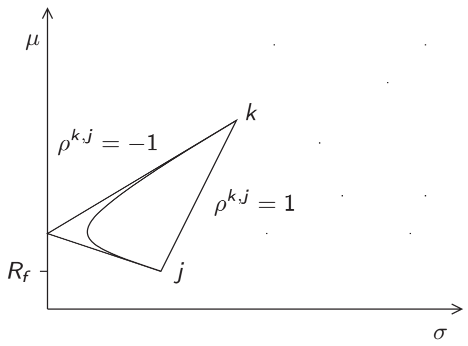
takes values between -1 (perfectly negatively correlated) and $+1$ (perfectly positively correlated).

Further explanation of possible correlation values can be found in the text book on page 97.

Correlation

When the risky assets are perfectly negatively correlated, i.e., when $\mathrm{corr}_{k,j} = -1$ the portfolio may even achieve an expected return higher than the riskfree rate without bearing additional risk.

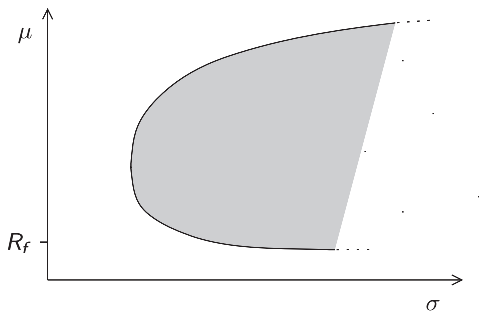
Diversification of two assets



Minimum-variance opportunity set

- Investors can build portfolios from risky and riskfree assets but also portfolios from other portfolios etc.
- The set of possible μ-σ-combinations offered by portfolios of risky assets that yield minimum variance for a given rate of return is called minimum-variance opportunity set or portfolio bound.

Minimum-variance opportunity set



Optimal investments

Choosing an optimal portfolio is to pick a portfolio with the highest expected returns for a given level of risk. This is similar to the following optimization problem:

$$\min_{\lambda_k,\lambda_j} \sum_k \sum_j \lambda_k \, \mathrm{cov}_{k,j} \, \lambda_j$$

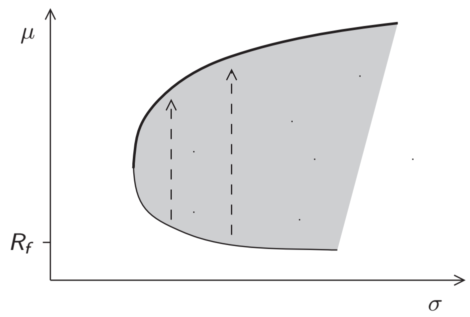
$$\text{such that } \sum_k \lambda_k \mu_k = \mathrm{const} \text{ and } \sum_k \lambda_k = 1,$$

where λ_k denote the proportion of money invested in asset k.

Efficient Frontier

- Solution of optimization problem gives the mean-variance opportunity set or the portfolio bound.
- Efficient portfolios focus on that part of the mean-variance efficient set that is not dominated by lower risk and higher return: upper part of the portfolio bound.

Efficient Frontier



Optimal Portfolio

- If an investor combines a risky asset (or a portfolio of risky assets) with a riskless security, he must choose a point on the line connecting both assets.
- This is a straight line, since covariance is zero and therefore standard deviation σ_λ is a *linear* function of the portfolio weights.

Capital Market Line

- Best portfolio combination: when the line achieves its highest possible slope.
- Defines the Capital Market Line (CML).
- Its slope is called Sharpe ratio, $(\mu_\lambda - R_f)/\sigma_\lambda$.
- Point at which the CML touches the efficient frontier is the tangent portfolio.

Optional Excursion: Mathematical Analysis of the Minimum-Variance Opportunity Set*

- It is sometimes said that the minimum-variance opportunity set is convex.
- This is not always the case: In the case of two assets, the opportunity set is *only* convex if their correlation is $+1$.
- We don't need convexity to prove existence of a tangent portfolio, we only need that the opportunity set is closed and certain properties of the efficient frontier that we summarize later.

Opportunity set is closed and connected

Lemma

If we have finitely many assets, the minimum-variance opportunity set is closed and connected.

The proof can be found in the text book on page 101f.

Infinitely many assets

What about if we have infinitely many assets? In this case the opportunity set does not have to be closed.

Example

Perfectly correlated assets with $\mu_k = 1 - 1/k$ and $\sigma_k = 1$. The opportunity set is given by $\{(\mu, 1) | \mu \in [0, 1)\}$ and is obviously not closed.

We better stick to the case of finitely many assets.

Efficient frontier can be discontinuous

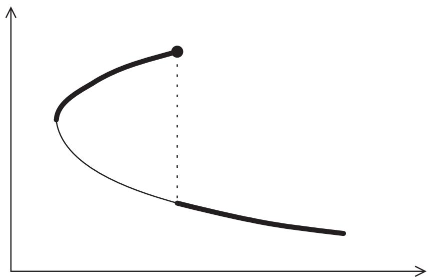
Lemma

If we have finitely many assets, the efficient frontier can be described as the graph of a function $f : [a, b]$*, where* $0 \leq a \leq b < \infty$*. Moreover there exists a point* $c \in [a, b]$ *such that* f *is concave and increasing on* $[a, c]$ *and decreasing on* $[c, b]$*.*

The proof can be found in the text book on page 102.

Efficient frontier can be discontinuous

Example: two assets, correlation less than 1.



Existence of a tangent portfolio

Proposition

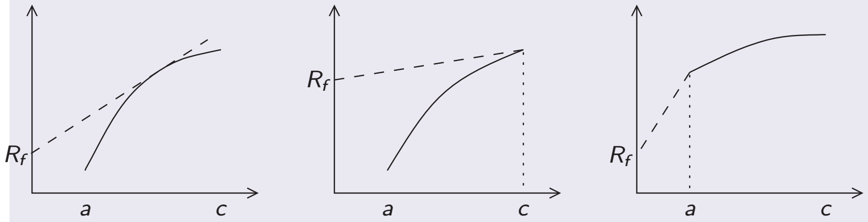
If we have finitely many assets, and at least one asset has a mean which is not lower than the return R_f of the risk-free asset, then a tangent portfolio exists.

The proof can be found in the text book on page 102.

Existence of a tangent portfolio

Proof.

Now, we have to distinguish three cases:



In all three cases, the constructed line cannot lie below other points of the efficient frontier, since f is decreasing for values larger than c, but the tangent line is increasing (or at least horizontal), since $f(c) \geq R_f$. □

Two-Fund Separation Theorem (1)

- Optimal asset allocation of risky assets and a riskless security depends on investor's preferences

$$U^i(\mu_\lambda, \sigma_\lambda^2) := \mu_\lambda - \frac{\rho^i}{2}\sigma_\lambda^2,$$

  where ρ^i is a risk aversion parameter of investor i. The higher this parameter, the higher is the slope of the utility function.
- The higher the risk aversion, the higher is the required expected return for a unit risk (required risk premium).
- Different investors have different risk-return preferences. Investors with higher (lower) level of risk aversion choose portfolios with a low (high) level of expected return and variance, i.e., their portfolios move down (up) the efficient frontier.

Two-Fund Separation Theorem (2)

The *Separation Theorem* of Tobin (1958) states that agents should diversify between the risk free asset (e.g., money) and a single optimal portfolio of risky assets.
Different attitudes toward risk result in different combinations of the risk free asset and the optimal portfolio of risky assets.

- More conservative investors will choose to put a higher fraction of their wealth into the risk free asset
- more aggressive investors decide to borrow capital on the money market and invest it in the Tangent Portfolio

This property is known as Two-Fund Separation.

Computing the Tangent Portfolio (1)

According to the Two-Fund Separation an investor with utility

$$U^i(\mu_{\boldsymbol{\lambda}}, \sigma^2_{\boldsymbol{\lambda}}) = \mu_{\boldsymbol{\lambda}} - \frac{\rho^i}{2}\sigma^2_{\boldsymbol{\lambda}}$$

has to decide how to split his wealth between the optimal portfolio of risky assets with a certain variance-covariance structure (Tangent Portfolio) and the riskless asset.

Further information on how to compute the tangent portfolio can be found in the text book on page 106f.

Market Equilibrium

Market Equilibrium

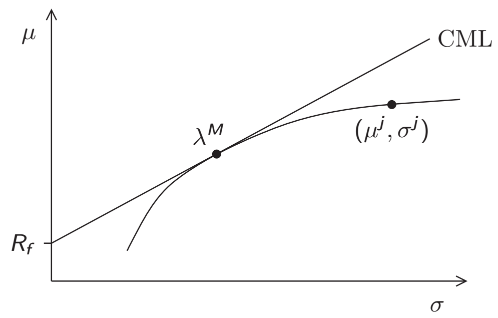
We want to study market equilibria, therefore we make the following observation:
If individual portfolios satisfy the Two-Fund-Separation, then by setting demand equal to supply the sum of the individual portfolios must be proportional to the vector of market capitalization $\boldsymbol{\lambda}^M$:

$$\sum_i \lambda_k^i = \Big(\sum_i (1 - \lambda_0^i)\Big)\lambda_k^T = \lambda_k^M.$$

Hence, in equilibrium, the normalized *Tangent Portfolio* will be identical to the *Market Portfolio*.

Derivation of the SML (1)

Compare the slopes of the Capital Market Line and a curve j that is obtained by mixing a portfolio of any asset j with the market portfolio. By the tangency property of $\boldsymbol{\lambda}^M$ these two slopes must be equal!



Derivation of the SML (2)

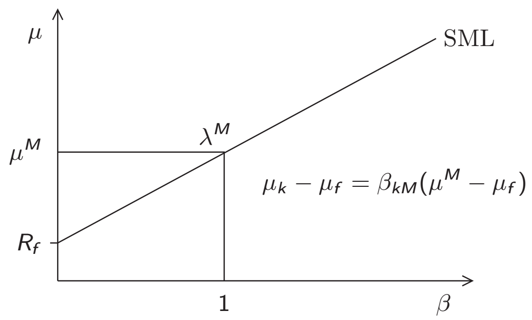
The slope of the Capital Market Line is

$$\frac{R_f - \mu^M}{-\sigma_M}.$$

A detailed derivation of the SML can be found in the text book on page 107f.

Derivation of the SML (3)

The result is the Security Market Line:



Derivation of the SML (4)

The difference to the mean-variance analysis is the risk measure:

- In the CAPM the asset's risk is captured by the factor β instead of the standard deviation of asset's returns.
- It measures the sensitivity of asset j returns to changes in the returns of the market portfolio. This is the so called systematic risk.

Market Neutral Strategies

The Capital Asset Pricing Model has many applications for investment managers and corporate finance.
Example: *Market Neutral Strategy* followed by some hedge funds.

- This strategy aims a zero exposure to market risk.
- To exclude the impact of market movements, it takes simultaneous long and short positions on risky assets.
- These assets have the same Beta (as measure for market risk) but different market prices.
- Under the assumption that market prices will eventually return to their fundamental value defined by the CAPM, hedge fund managers take long positions in underpriced assets and short positions in overpriced assets.

We will discuss later the potential risks of this strategy.

Empirical Validity of the CAPM

- An advisor should apply the same expectations when giving recommendations to different clients
- Hence, following the two-fund separation property, he should recommend the same portfolio of risky assets.
- Canner, Mankiw and Weil [Canner et al., 1997] showed that this simple rule is however not followed by advisors.
- An application example of the CAPM for investment managers can be found in the text book on page 108f.
- Further empirical validity of the CAPM can be found in the text book on page 109 f.

Empirical Validity of the CAPM

- One of the nice properties of the SML is that it suggests a linear relation between the Beta and the excess returns.
- Many studies found that market risk, the Beta, indeed explains the excess returns of assets. But more factors are needed to get a really good fit.
- Most famous additional factors are *value*, *size* and *momentum*. Investing in value stocks give significantly higher returns – even with lower Beta – than investing in glamour stocks.
- Also, investing in small cap stocks has this feature.
- Finally, investing in stocks that have gone up is increasing returns in the short run and the reverse is true in the long run. [Fama and French, 1992], [Fama and French, 1998] and [Lakonishok et al., 1994]