

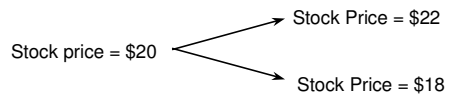
Binomial Trees

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A Simple Binomial Model

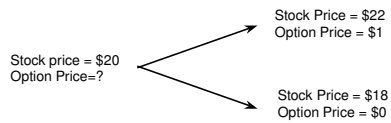
- A stock price is currently \$20
- In 3 months it will be either \$22 or \$18



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A Call Option

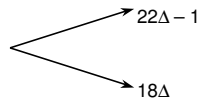
A 3-month call option on the stock has a strike price of 21.



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Setting Up a Riskless Portfolio

- For a portfolio that is long Δ shares and a short 1 call option values are



- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

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Valuing the Portfolio

(Risk-Free Rate is 12%)

- The riskless portfolio is:
 - long 0.25 shares
 - short 1 call option
- The value of the portfolio in 3 months is $22 \times 0.25 - 1 = 4.50$
- The value of the portfolio today is $4.5e^{-0.12 \times 0.25} = 4.3670$

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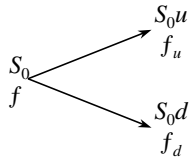
Valuing the Option

- The portfolio that is
 - long 0.25 shares
 - short 1 optionis worth 4.367
- The value of the shares is 5.000 (= 0.25×20)
- The value of the option is therefore 0.633 ($5.000 - 0.633 = 4.367$)

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Generalization

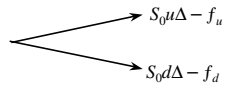
A derivative lasts for time T and is dependent on a stock



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Generalization (continued)

- Value of a portfolio that is long Δ shares and short 1 derivative:



- The portfolio is riskless when $S_0u\Delta - f_u = S_0d\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d}$$

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Generalization (continued)

- Value of the portfolio at time T is $S_0u\Delta - f_u$
- Value of the portfolio today is $(S_0u\Delta - f_u)e^{-rT}$
- Another expression for the portfolio value today is $S_0\Delta - f$
- Hence

$$f = S_0\Delta - (S_0u\Delta - f_u)e^{-rT}$$

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Generalization

(continued)

Substituting for Δ we obtain

$$f = [pf_u + (1-p)f_d]e^{-rT}$$

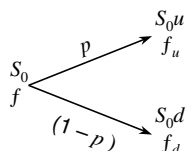
where

$$p = \frac{e^{rT} - d}{u - d}$$

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p as a Probability

- It is natural to interpret p and $1-p$ as probabilities of up and down movements
- The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate



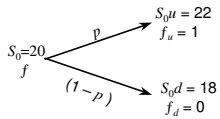
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Risk-Neutral Valuation

- When the probability of an up and down movements are p and $1-p$ the expected stock price at time T is S_0e^{rT}
- This shows that the stock price earns the risk-free rate
- Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate
- This is known as using risk-neutral valuation

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Original Example Revisited



p is the probability that gives a return on the stock equal to the risk-free rate:

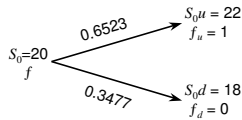
$$20e^{0.12 \times 0.25} = 22p + 18(1-p) \text{ so that } p = 0.6523$$

Alternatively:

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

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Valuing the Option Using Risk-Neutral Valuation



The value of the option is

$$e^{-0.12 \times 0.25} (0.6523 \times 1 + 0.3477 \times 0) = 0.633$$

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Irrelevance of Stock's Expected Return

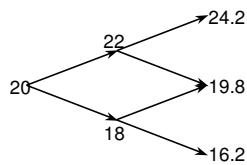
- When we are valuing an option in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant
- This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant

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TNLPJ18

- $S_0=18.70$ (09/26/2011), matures on 10/19/2011 (15 business days $T=15/252$).
- Assume that in 15 b.d. Telemar TNLP4 can go up to 20 or down to 16. The SELIC rate is 12%.
- Find the option price

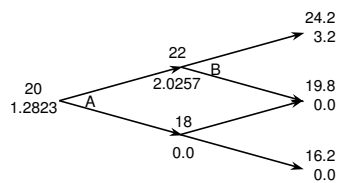
A Two-Step Example



- $K=21$, $r = 12\%$
- Each time step is 3 months

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Valuing a Call Option

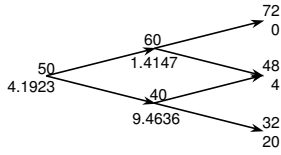


Value at node B
 $= e^{-0.12 \times 0.25} (0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$

Value at node A
 $= e^{-0.12 \times 0.25} (0.6523 \times 2.0257 + 0.3477 \times 0) = 1.2823$

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A Put Option Example



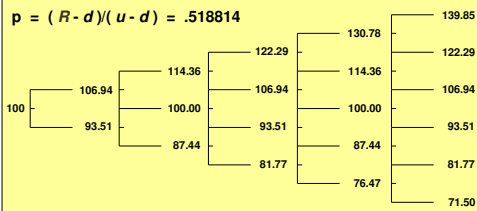
$K = 52, T=2, \text{time step} = 1\text{yr}$
 $r = 5\%, u = 1.2, d = 0.8, p = 0.6282$

Call Example

$S = 100, t = .25, r = 0.10$
 $K = 0, \sigma = .3$
 $n = 5$

$u \equiv e^{\sigma\sqrt{t/n}} = 1.06938$
 $d \equiv 1/u = .935118$
 $R \equiv e^{rt/n} = 1.005$

$p = (R - d)/(u - d) = .518814$

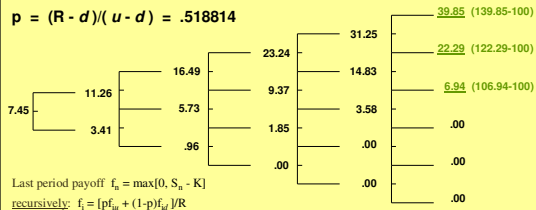


Call Example

$S = 100, t = .25, r = 0.10$
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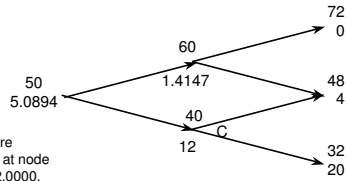
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Last period payoff $f_n = \max[0, S_n - K]$
 recursively: $f_i = [pf_{i+1} + (1-p)f_{i+2}]/R$

What Happens When the Put Option is American



The American feature increases the value at node C from 9.4636 to 12.0000.

This increases the value of the option from 4.1923 to 5.0894.

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Choosing u and d

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where σ is the volatility and Δt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein

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Assets Other than Non-Dividend Paying Stocks

- For options on stock indices, currencies and futures the basic procedure for constructing the tree is the same except for the calculation of p

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The Probability of an Up Move

$$p = \frac{a - d}{u - d}$$

$a = e^{r_f \Delta t}$ for a nondividend paying stock

$a = e^{(r - q) \Delta t}$ for a stock index where q is the dividend yield on the index

$a = e^{(r - r_f) \Delta t}$ for a currency where r_f is the foreign risk - free rate

$a = 1$ for a futures contract

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Binomial Trees for n steps

$$c = e^{-r_f t} \sum_{j=0}^n \frac{n!}{(n-j)! j!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$$

Option is in the money when $j > \alpha$ where

$$\alpha = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$$

so that

$$c = e^{-r_f t} (S_0 U_1 - K U_2)$$

where

$$U_1 = \sum_{j=\alpha}^n \frac{n!}{(n-j)! j!} p^j (1-p)^{n-j} u^j d^{n-j}$$

$$U_2 = \sum_{j=0}^{\alpha} \frac{n!}{(n-j)! j!} p^j (1-p)^{n-j}$$

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Binomial Trees for n Large

- The expression for U_1 can be written

$$U_1 = [pu + (1-p)d]^n \sum_{j=\alpha}^n \frac{n!}{(n-j)! j!} (p^*)^j (1-p^*)^{n-j} = e^{r_f t} \sum_{j=\alpha}^n \frac{n!}{(n-j)! j!} (p^*)^j (1-p^*)^{n-j}$$

where $p^* = \frac{pu}{pu + (1-p)d}$

- Both U_1 and U_2 can now be evaluated in terms of the cumulative binomial distribution
- We now let the number of time steps tend to infinity and use the result that a binomial distribution tends to a normal distribution

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