# Binomial Trees

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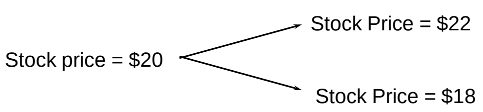
## A Simple Binomial Model

- A stock price is currently $20
- In 3 months it will be either $22 or $18

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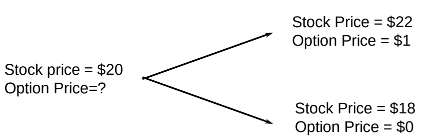
## A Call Option

A 3-month call option on the stock has a strike price of 21.

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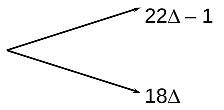
## Setting Up a Riskless Portfolio

- For a portfolio that is long Δ shares and a short 1 call option values are




- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

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## Valuing the Portfolio
### (Risk-Free Rate is 12%)

- The riskless portfolio is:
  - long 0.25 shares
  - short 1 call option
- The value of the portfolio in 3 months is
  - $22 \times 0.25 - 1 = 4.50$
- The value of the portfolio today is
  - $4.5e^{-0.12\times0.25} = 4.3670$

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## Valuing the Option

- The portfolio that is
  - long 0.25 shares
  - short 1 option

  is worth 4.367
- The value of the shares is
  - 5.000 (= 0.25 × 20 )
- The value of the option is therefore
  - 0.633 ( 5.000 – 0.633 = 4.367 )

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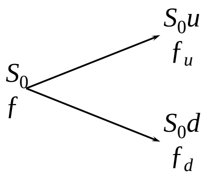
## Generalization

A derivative lasts for time $T$ and is dependent on a stock

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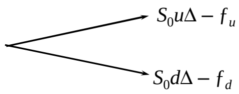
## Generalization (continued)

- Value of a portfolio that is long $\Delta$ shares and short 1 derivative:




- The portfolio is riskless when $S_0u\Delta - f_u = S_0d\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d}$$

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## Generalization (continued)

- Value of the portfolio at time $T$ is $S_0u\Delta - f_u$
- Value of the portfolio today is $(S_0u\Delta - f_u)e^{-rT}$
- Another expression for the portfolio value today is $S_0\Delta - f$
- Hence

$$f = S_0\Delta - (S_0u\Delta - f_u)e^{-rT}$$

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## Generalization
(continued)

Substituting for Δ we obtain

$$f = [\, pf_u + (1-p)f_d \,]e^{-rT}$$

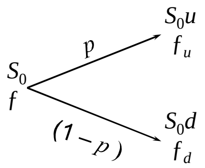
where

$$p = \frac{e^{rT} - d}{u - d}$$



## *p* as a Probability

- It is natural to interpret $p$ and 1-$p$ as probabilities of up and down movements
- The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate

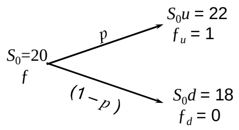





## Risk-Neutral Valuation

- When the probability of an up and down movements are $p$ and 1-$p$ the expected stock price at time $T$ is $S_0e^{rT}$
- This shows that the stock price earns the risk-free rate
- Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate
- This is known as using risk-neutral valuation

# Original Example Revisited




*p* is the probability that gives a return on the stock equal to the risk-free rate:

$20e^{0.12 \times 0.25} = 22p + 18(1 - p)$ so that $p = 0.6523$

Alternatively:

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

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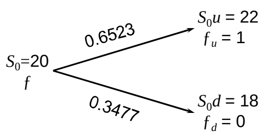
# Valuing the Option Using Risk-Neutral Valuation




The value of the option is

$e^{-0.12 \times 0.25}$ (0.6523×1 + 0.3477×0)

= 0.633

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# Irrelevance of Stock's Expected Return

- When we are valuing an option in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant
- This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant

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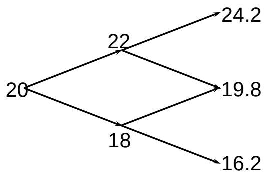
# A Two-Step Example




- $K$=21, $r$ = 12%
- Each time step is 3 months

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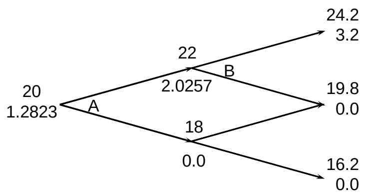
# Valuing a Call Option




Value at node B

$= e^{-0.12\times0.25}(0.6523\times3.2 + 0.3477\times0) = 2.0257$

Value at node A

$= e^{-0.12\times0.25}(0.6523\times2.0257 + 0.3477\times0) = 1.2823$

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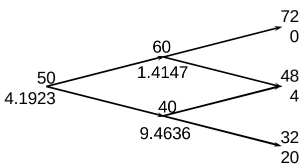
# A Put Option Example




$K$ = 52, T=2, time step =1yr

$r$ = 5%, $u$ =1.2, $d$ = 0.8, $p$ = 0.6282

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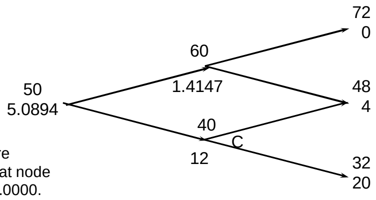
## What Happens When the Put Option is American




The American feature increases the value at node C from 9.4636 to 12.0000.

This increases the value of the option from 4.1923 to 5.0894.

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## Choosing u and d

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where $\sigma$ is the volatility and $\Delta t$ is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein

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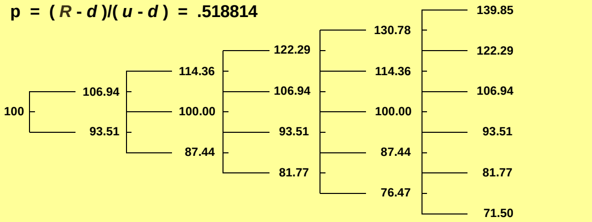
## Call Example

S = 100 t = .25 r = 0.10
K = 0 σ = .3
n = 5

$u \equiv e^{\sigma\sqrt{t/n}} = 1.06938$
$d \equiv 1/u = .935118$
$R \equiv e^{rt/n} = 1.005$

p = ( *R* - *d* )/( *u* - *d* ) = .518814

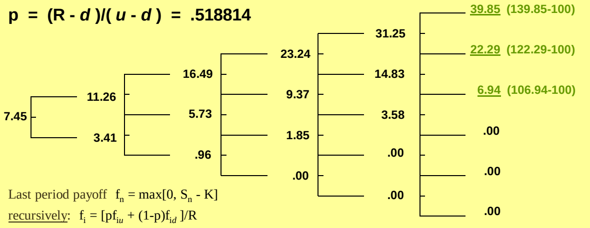


## Call Example

S = 100   t = .25   r = 0.10
K = 100   $\sigma$ = .3
n = 5

$u \equiv e^{\sigma\sqrt{t/n}} = 1.06938$
$d \equiv 1/u = .935118$
$R \equiv e^{r\,t/n} = 1.005$

p = (R - *d*)/(*u* - *d*) = .518814




Last period payoff $f_n = \max[0, S_n - K]$
recursively: $f_i = [pf_{iu} + (1-p)f_{id}]/R$

## Assets Other than Non-Dividend Paying Stocks

- For options on stock indices, currencies and futures the basic procedure for constructing the tree is the same except for the calculation of *p*

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## The Probability of an Up Move

$$p = \frac{a - d}{u - d}$$

$a = e^{r\Delta t}$ for a nondividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where $q$ is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where $r_f$ is the foreign risk - free rate

$a = 1$ for a futures contract

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## Binomial Trees for *n* steps

$$c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{(n-j)!\, j!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K,\ 0)$$

Option is in the money when $j > \alpha$ where

$$\alpha = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$$

so that

$$c = e^{-rT}(S_0 U_1 - K U_2)$$

where

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!\, j!} p^j (1-p)^{n-j} u^j d^{n-j}$$

$$U_2 = \sum_{j>\alpha} \frac{n!}{(n-j)!\, j!} p^j (1-p)^{n-j}$$

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## Binomial Trees for *n* Large

- The expression for $U_1$ can be written

$$U_1 = [pu + (1-p)d]^n \sum_{j>\alpha} \frac{n!}{(n-j)!\, j!} (p^*)^j (1-p^*)^{n-j} = e^{rT} \sum_{j>\alpha} \frac{n!}{(n-j)!\, j!} (p^*)^j (1-p^*)^{n-j}$$

where $p^* = \dfrac{pu}{pu + (1-p)d}$

- Both $U_1$ and $U_2$ can now be evaluated in terms of the cumulative binomial distribution
- We now let the number of time steps tend to infinity and use the result that a binomial distribution tends to a normal distribution

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