The Black-Scholes-Merton Model

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Fundação Getulio Vargas

Nobel Prize 1997

- Merton, R.C.: " Theory of Rational Option Pricing", *Bell Jounal of Economics and Management Science,* 4(1973), 141-183
- Black, F., and M. Scholes,: " The Pricing of Options and Corporate Liabilities", *Journal of Political Economy,* 81(1973), 637-659

Return Rates

$$R_t = \log \frac{S_t}{S_{t-1}} \approx \frac{S_t}{S_{t-1}} - 1,$$

Where S_t and R_t are the stock price and the return rate at time t, respectively. Now denote the rhs by

$$R_t^* = \frac{S_t}{S_{t-1}} - 1,$$

Stock Price Models

- Bachelier (1900)

$$S_t = S_0 (X_t - X_{t-1})$$

- Samuelson (1964)

$$S_t = S_0 e^{X_t}$$

In both cases X_t is a random variable Normally distributed

Advantages

- Computing the return rate we have*:*

$$R_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \log\left(\frac{S_0 e^{X_t}}{S_0 e^{X_{t-1}}}\right) = X_t - X_{t-1}$$

From here, return rate will be also a random variable Normally distributed

The Stock Price Assumption

- Consider a stock whose price is *S*
- In a short period of time of length *Δt,* the return on the stock is normally distributed:

$$\frac{\Delta S}{S} \approx N\left(\mu \Delta t, \sigma^2 \Delta t\right)$$

where μ is expected return and σ is volatility

$$\Delta S = \mu S \Delta t + \sigma S \Delta \varepsilon, \text{ onde } \Delta \varepsilon \text{ é } N(0, \sqrt{\Delta t})$$

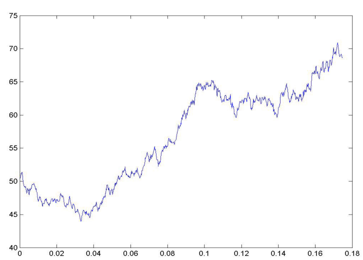
The Stock Price Assumption

$$R_{t+\Delta t} = \frac{S_{t+\Delta t} - S_t}{S_t} = \frac{\Delta S}{S} = \mu \Delta t + \sigma \Delta \varepsilon,$$

- Return has two components, one deterministic and other random!
- When $\Delta t \rightarrow 0$ we have a *Stochastic Differential Equation*:

$$dS = \mu S dt + \sigma S dB_t, \text{ onde } dB_t \text{ é } N(0, \sqrt{dt})$$

Simulating the Stock Price Model



Solution of SDE

- The solution of the SDE with initial condition S_0 is given by

$$S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma B_T}$$

The Lognormal Property

- It follows from this assumption that :

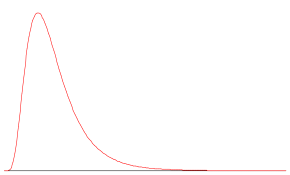
$$\ln S_T - \ln S_0 \approx N\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

or

$$\ln S_T \approx N\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

- Since the logarithm of S_T is normal, S_T is lognormally distributed

The Lognormal Distribution



$$E(S_T) = S_0 e^{\mu T}$$

$$Var(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

The Expected Return

- The expected value of the stock price is $S_0e^{\mu T}$
- The expected return on the stock is $\mu - \sigma^2/2$ not μ

This is because

$\ln[E(S_T/S_0)]$ and $E[\ln(S_T/S_0)]$

are not the same

μ and $\mu-\sigma^2/2$

Suppose we have daily data for a period of several months

1. μ is the average of the returns in each day [$=E(\Delta S/S)$]
2. $\mu-\sigma^2/2$ is the expected return over the whole period covered by the data measured with continuous compounding (or daily compounding, which is almost the same)

Mutual Fund Returns

- Suppose that returns in successive years are 15%, 20%, 30%, -20% and 25%
- The arithmetic mean of the returns is 14%
- The returned that would actually be earned over the five years (the geometric mean) is 12.4%

The Volatility

- The volatility is the standard deviation of the continuously compounded rate of return in 1 year
- The standard deviation of the return in time Δt is $\sigma\sqrt{\Delta t}$
- If a stock price is $50 and its volatility is 25% per year what is the standard deviation of the price change in one day?

Estimating Volatility from Historical Data

1. Take observations $S_0, S_1, \ldots, S_n$ at intervals of τ years
2. Calculate the continuously compounded return in each interval as:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

3. Calculate the standard deviation, s , of the u_i´s
4. The historical volatility estimate is: $\hat{\sigma} = \frac{s}{\sqrt{\tau}}$

Nature of Volatility

- Volatility is usually much greater when the market is open (i.e. the asset is traded) than when it is closed
- For this reason time is usually measured in "trading days" not calendar days when options are valued

The Concepts Underlying Black-Scholes

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes differential equation

The Derivation of the Black-Scholes Differential Equation

$$\Delta S = \mu S\,\Delta t + \sigma S\,\Delta z \qquad (1)$$

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S\,\Delta z \ldots.(2)$$

We set up a portfolio consisting of

-1: derivative

$+\frac{\partial f}{\partial S}$: shares

The Derivation of the Black-Scholes Differential Equation continued

The value of the portfolio Π is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time Δt is given by

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$$

The Derivation of the Black-Scholes Differential Equation continued

The return on the portfolio must be the risk-free rate. Hence

$$\Delta \Pi = r\,\Pi\,\Delta t$$

We substitute for Δf and ΔS in these equations to get the Black-Scholes differential equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

The Differential Equation

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- In the European call and put case, we have the boundary conditions:

  $f(S_T)=max\{S_T-X,0\}$ *ou* $f(S_T)=max\{X-S_T,0\}$
- In these cases we can obtain explicit solutions.

The Black-Scholes Formulas

$$c = S_0\, N(d_1) - K\, e^{-rT} N(d_2)$$
$$p = K\, e^{-rT}\, N(-d_2) - S_0\, N(-d_1)$$

where $d_1 = \dfrac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The N(x) Function

- $N(x)$ is the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 is less than x
- See tables at the end of the book

Properties of Black-Scholes Formula

- As S_0 becomes very large c tends to $S_0 - Ke^{-rT}$ and p tends to zero
- As S_0 becomes very small c tends to zero and p tends to $Ke^{-rT} - S_0$

Risk-Neutral Valuation

- The variable µ does not appearin the Black-Scholes equation
- The equation is independent of all variables affected by risk preference
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- This leads to the principle of risk-neutral valuation

Applying Risk-Neutral Valuation

1. Assume that the expected return from the stock price is the risk-free rate
2. Calculate the expected payoff from the option
3. Discount at the risk-free rate

$$f = E^Q(e^{-rT} f(S_T))$$

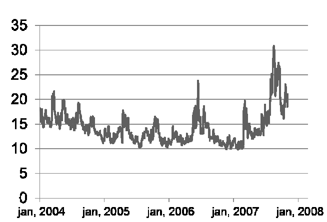
Parameters

- To apply Black and Scholes formula we need to estimate the unobseved parameters.
- Volatility: Historical or Implied

Implied Volatility

- The implied volatility of an option is the volatility for which the Black-Scholes price equals the market price
- There is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

The VIX S&P500 Volatility Index



Chapter 24 explains how the index is calculated

Option Price

- Data:
  Stripe Price (k) = 56,00
  Sport Price (S) = 54,90
  Interest rate (i) = 0,11 % daily or 0,0011
  Volatility (σ) = 40,00 % or 0,4000
  Maturity (n) = 44 days
- Compute the option price

Exemplo

- Obntain *r=252*ln(1+i)*
- *Then r=252*ln(1,0011)=0,277*
- *d1=(ln(54,9/56)+(0,277+0.4^2/2)*44/252)./(0.4*(44/252)^1/2)*
  =0.254294
- *d2=d1-0.4*(44/252)^1/2=0.087152*
- *N(d1)=0,6*
- *N(d2)=0,5347*
- *C=54,9*N(d1)-56*e^(-0.277*44/252)*N(d2)=**4,4295***

Implied Volatility

- VALED30
- So=27,05
- R=ln(1,1125)
- T=21/252
- X=30
- cmax=0,79 ,cmin=0,59
- Sigma=?

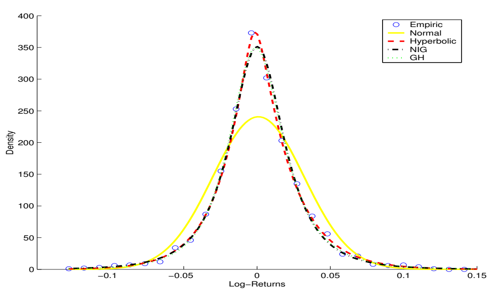
Limitations of B&S Model

Log-Normality

- Empirical Evidence: Stock prices present many outliers; Returns are leptokurticos (Mandelbrot [1963]).
- Volatility is not constant over time

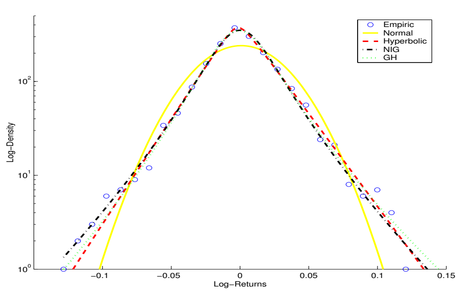
Problems with B&S

Vale 5

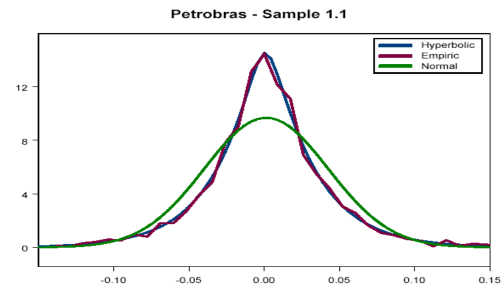


Problems with B&S

Vale 5



Problems with B&S



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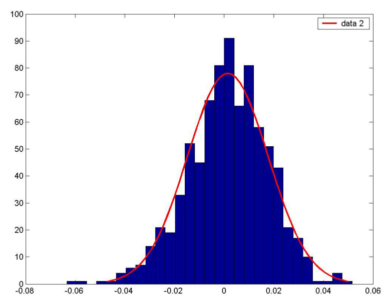

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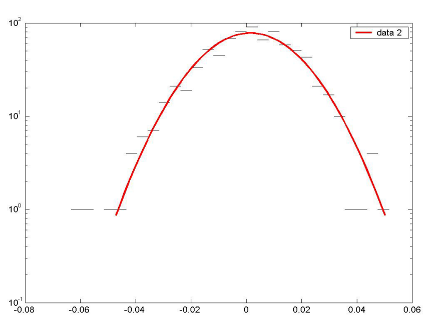

Limitations of B&S Model

Frictionless Market

- Transaction Cost
- Tnterest rate spread
- Asymmetric Information
- Short sales restrictions
- Liquidity

Exercices

1) Spot price is 42, the strike price of a 6-month European call is 40, if the risk-free interest rate is 10% p.y. and volatility is 20%. Find the price of the option.

Exercícios

2) Under the LogNormality assumption, find the price of a derivative whose payoff is $1000 in 1 month is stock price be greater than $45 and zero otherwise. Additionally, we know that spot price is $40, volatility is 30% and risk-free interest rate is 9% p.y.