Binomial Trees

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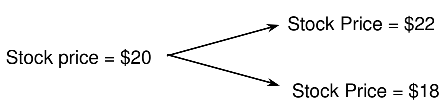
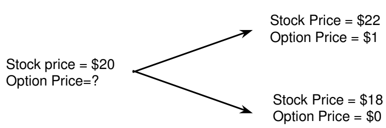
A Simple Binomial Model

- A stock price is currently $20
- In 3 months it will be either $22 or $18



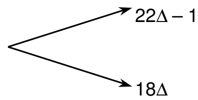
A Call Option

A 3-month call option on the stock has a strike price of 21.



Setting Up a Riskless Portfolio

- For a portfolio that is long Δ shares and a short 1 call option values are



- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Valuing the Portfolio
(Risk-Free Rate is 12%)

- The riskless portfolio is:
  - long 0.25 shares
  - short 1 call option
- The value of the portfolio in 3 months is
  - $22 \times 0.25 - 1 = 4.50$
- The value of the portfolio today is
  - $4.5e^{-0.12\times 0.25} = 4.3670$

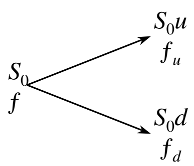
Valuing the Option

- The portfolio that is
  - long 0.25 shares
  - short 1 option

  is worth 4.367
- The value of the shares is
  - 5.000 (= 0.25 × 20)
- The value of the option is therefore
  - 0.633 (5.000 – 0.633 = 4.367)

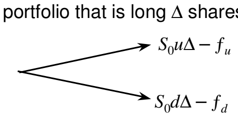
Generalization

A derivative lasts for time T and is dependent on a stock



Generalization *(continued)*

- Value of a portfolio that is long Δ shares and short 1 derivative:



- The portfolio is riskless when $S_0u\Delta - f_u = S_0d\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d}$$

Generalization *(continued)*

- Value of the portfolio at time T is $S_0u\Delta - f_u$
- Value of the portfolio today is $(S_0u\Delta - f_u)e^{-rT}$
- Another expression for the portfolio value today is $S_0\Delta - f$
- Hence

$$f = S_0\Delta - (S_0u\Delta - f_u)e^{-rT}$$

Generalization
(continued)

Substituting for Δ we obtain

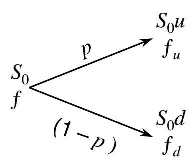
$$f = [\, p f_u + (1-p) f_d \,] e^{-rT}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

p *as a Probability*

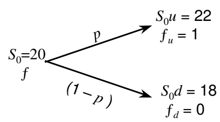
- It is natural to interpret p and 1-p as probabilities of up and down movements
- The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate



Risk-Neutral Valuation

- When the probability of an up and down movements are p and 1-p the expected stock price at time T is S_0e^{rT}
- This shows that the stock price earns the risk-free rate
- Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate
- This is known as using risk-neutral valuation

Original Example Revisited



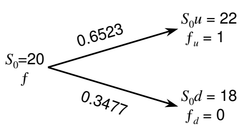
p is the probability that gives a return on the stock equal to the risk-free rate:

$20e^{0.12 \times 0.25} = 22p + 18(1 - p)$ so that $p = 0.6523$

Alternatively:

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing the Option Using Risk-Neutral Valuation



The value of the option is

$e^{-0.12 \times 0.25}(0.6523 \times 1 + 0.3477 \times 0)$

$= 0.633$

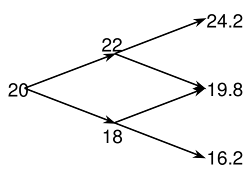
Irrelevance of Stock's Expected Return

- When we are valuing an option in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant
- This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant

TNLPJ18

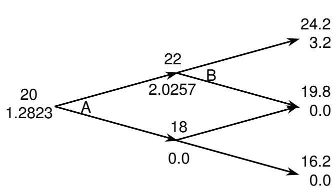
- S0=18.70 (09/26/2011), matures on 10/19/2011 (15 business days T=15/252).
- Assume that in 15 b.d. Telemar TNLP4 can go up to 20 or down to 16. The SELIC rate is 12%.
- Find the option price

A Two-Step Example



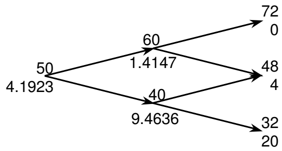
- K=21, r = 12%
- Each time step is 3 months

Valuing a Call Option



Value at node B
$= e^{-0.12\times0.25}(0.6523\times3.2 + 0.3477\times0) = 2.0257$
Value at node A
$= e^{-0.12\times0.25}(0.6523\times2.0257 + 0.3477\times0) = 1.2823$

A Put Option Example



K = 52, time step =1yr

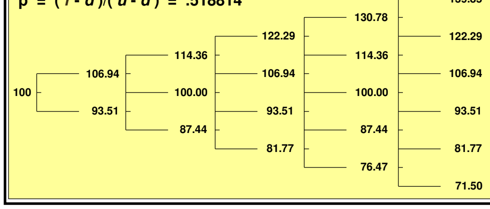
r = 5%, u =1.32, d = 0.8, p = 0.6282

Call Example

S = 100 t = .25 r = 0.10
K = <u>0</u> σ = .3
n = 5

$u \equiv e^{\sigma\sqrt{t/n}}$ = 1.06938
$d \equiv 1/u$ = .935118
R ≡ $1.10^{t/n}$ = 1.00478

p = (r - d)/(u - d) = .518814



Call Example

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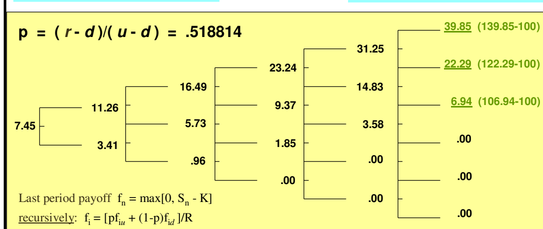
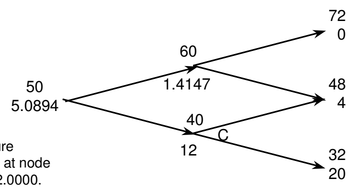


Last period payoff $f_n = \max[0, S_n - K]$
<u>recursively</u>: $f_i = [pf_{iu} + (1-p)f_{id}]/R$

What Happens When the Put Option is American



The American feature increases the value at node C from 9.4636 to 12.0000.

This increases the value of the option from 4.1923 to 5.0894.

Choosing u and d

One way of matching the volatility is to set

$$u = e^{\sigma \sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma \sqrt{\Delta t}}$$

where σ is the volatility and Δt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein

Assets Other than Non-Dividend Paying Stocks

- For options on stock indices, currencies and futures the basic procedure for constructing the tree is the same except for the calculation of p

The Probability of an Up Move

$$p = \frac{a-d}{u-d}$$

$a = e^{r\Delta t}$ for a nondividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where q is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where r_f is the foreign risk - free rate

$a = 1$ for a futures contract

Binomial Trees for n *steps*

$$c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K,\ 0)$$

Option is in the money when $j > \alpha$ where

$$\alpha = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$$

so that

$$c = e^{-rT}(S_0 U_1 - K U_2)$$

where

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} u^j d^{n-j}$$

$$U_2 = \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j}$$

Binomial Trees for n *Large*

- The expression for U_1 can be written

$$U_1 = [pu + (1-p)d]^n \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} (p^*)^j (1-p^*)^{n-j} = e^{rT} \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} (p^*)^j (1-p^*)^{n-j}$$

where $p^* = \dfrac{pu}{pu + (1-p)d}$

- Both U_1 and U_2 can now be evaluated in terms of the cumulative binomial distribution
- We now let the number of time steps tend to infinity and use the result that a binomial distribution tends to a normal distribution