VaR for Options

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VaR of an Option

- Using the definition of Delta, VaR of a portfolio "P" with an option, can be calculated in the following way:

$$\Delta = \frac{\delta P}{\delta S}$$

or

$$\delta P = \Delta\, \delta S \qquad (16.3)$$

where δS is the dollar change in the stock price in one day and δP is, as usual, the dollar change in the portfolio in one day. We define δx as the percentage change in the stock price in one day, so that

$$\delta x = \frac{\delta S}{S}$$

It follows that an approximate relationship between δP and δx is

$$\delta P = S\Delta\, \delta x$$

With Many Options

When we have a position in several underlying market variables that includes options, we can derive an approximate linear relationship between δP and the δx_i's similarly. This relationship is

$$\delta P = \sum_{i=1}^{n} S_i \Delta_i\, \delta x_i \qquad (16.4)$$

where S_i is the value of the ith market variable and Δ_i is the delta of the portfolio with respect to the ith market variable. This corresponds to equation (16.1):

$$\delta P = \sum_{i=1}^{n} \alpha_i\, \delta x_i \qquad (16.5)$$

with $\alpha_i = S_i\, \Delta_i$. Equation (16.2) can therefore be used to calculate the standard deviation of δP.

Example

Example 16.1 A portfolio consists of options on Microsoft and AT&T. The options on Microsoft have a delta of 1,000, and the options on AT&T have a delta of 20,000. The Microsoft share price is $120, and the AT&T share price is $30. From equation (16.4), it is approximately true that

$$\delta P = 120 \times 1{,}000 \times \delta x_1 + 30 \times 20{,}000 \times \delta x_2$$

or

$$\delta P = 120{,}000\delta x_1 + 600{,}000\delta x_2$$

where δx_1 and δx_2 are the returns from Microsoft and AT&T in one day and δP is the resultant change in the value of the portfolio. (The portfolio is assumed to be equivalent to an investment of $120,000 in Microsoft and $600,000 in AT&T.) Assuming that the daily volatility of Microsoft is 2% and the daily volatility of AT&T is 1%, and the correlation between the daily changes is 0.3, the standard deviation of δP (in thousands of dollars) is

$$\sqrt{(120 \times 0.02)^2 + (600 \times 0.01)^2 + 2 \times 120 \times 0.02 \times 600 \times 0.01 \times 0.3} = 7.099$$

Because $N(-1.65) = 0.05$, the 5-day 95% value at risk is

$$1.65 \times \sqrt{5} \times 7,099 = \$26,193$$

With Gamma?

$$\delta P = \Delta\,\delta S + \tfrac{1}{2}\Gamma\,(\delta S)^2$$

is an improvement over the approximation in equation (16.3).[8] Setting

$$\delta x = \frac{\delta S}{S}$$

reduces this to

$$\delta P = S\Delta\,\delta x + \tfrac{1}{2}S^2\Gamma\,(\delta x)^2 \qquad (16.6)$$

The variable δP is not normal. Assuming that δx is normal with mean zero and standard deviation σ, the first three moments of δP are

$$E(\delta P) = \tfrac{1}{2}S^2\Gamma\sigma^2$$

$$E[(\delta P)^2] = S^2\Delta^2\sigma^2 + \tfrac{3}{4}S^4\Gamma^2\sigma^4$$

$$E[(\delta P)^3] = \tfrac{9}{2}S^4\Delta^2\Gamma\sigma^4 + \tfrac{15}{8}S^6\Gamma^3\sigma^6$$

What happen if we ignore this 3rd moment?

With 3rd Moment

Define μ_P and σ_P as the mean and standard deviation of δP, so that

$$\mu_P = E(\delta P)$$

$$\sigma_P^2 = E[(\delta P)^2] - [E(\delta P)]^2$$

The skewness, ξ_P, of the probability distribution of δP is defined as

$$\xi_P = \frac{1}{\sigma_P^3}E[(\delta P - \mu_P)^3] = \frac{E[(\delta P)^3] - 3E[(\delta P)^2]\mu_P + 2\mu_P^3}{\sigma_P^3}$$

Using the first three moments of δP, the Cornish–Fisher expansion estimates the qth percentile of the distribution of δP as

$$\mu_P + w_q\sigma_P$$

where

$$w_q = z_q + \tfrac{1}{6}(z_q^2 - 1)\xi_P$$

and z_q is the qth percentile of the standard normal distribution $\phi(0, 1)$.

Example

Example 16.2 Suppose that for a certain portfolio we calculate $\mu_P = -0.2$, $\sigma_P = 2.2$, and $\xi_P = -0.4$. If we assume that the probability distribution of δP is normal, the first percentile of the probability distribution of δP is

$$-0.2 - 2.33 \times 2.2 = -5.326$$

In other words, we are 99% certain that

$$\delta P > -5.326$$

When we use the Cornish–Fisher expansion to adjust for skewness and set $q = 0.01$, we obtain

$$w_q = -2.33 - \tfrac{1}{6}(2.33^2 - 1) \times 0.4 = -2.625$$

so that the first percentile of the distribution is

$$-0.2 - 2.625 \times 2.2 = -5.976$$

Taking account of skewness, therefore, changes the VaR from 5.326 to 5.976.

Cornish-Fisher Expansion

$$\Phi_X^{-1}(q) \approx \Phi_Z^{-1}(q) + \frac{\Phi_Z^{-1}(q)^2 - 1}{6}\kappa_3 + \frac{\Phi_Z^{-1}(q)^3 - 3\Phi_Z^{-1}(q)}{24}\kappa_4$$
$$- \frac{2\Phi_Z^{-1}(q)^3 - 5\Phi_Z^{-1}(q)}{36}\kappa_3^2 + \frac{\Phi_Z^{-1}(q)^4 - 6\Phi_Z^{-1}(q)^2 + 3}{120}\kappa_5$$
$$- \frac{\Phi_Z^{-1}(q)^4 - 5\Phi_Z^{-1}(q)^2 + 2}{24}\kappa_3\kappa_4 + \frac{12\Phi_Z^{-1}(q)^4 - 53\Phi_Z^{-1}(q)^2 + 17}{324}\kappa_3^3$$

$\mu_i = E(x-\mu)^i$, $k_i = \mu_i$, $i = 1,2,3$

$\kappa_4 = \mu_4 - 3\mu_2^2$

$\kappa_5 = \mu_5 - 10\mu_3\mu_2$

Add to 3rd List

16.17. Consider a position consisting of a \$300,000 investment in gold and a \$500,000 investment in silver. Suppose that the daily volatilities of these two assets are 1.8% and 1.2%, respectively, and that the coefficient of correlation between their returns is 0.6. What is the 10-day 97.5% value at risk for the portfolio? By how much does diversification reduce the VaR?

16.18. Consider a portfolio of options on a single asset. Suppose that the delta of the portfolio is 12, the value of the asset is \$10, and the daily volatility of the asset is 2%. Estimate the 1-day 95% VaR for the portfolio.

16.19. Suppose that the gamma of the portfolio in Problem 16.18 is −2.6. Derive a quadratic relationship between the change in the portfolio value and the percentage change in the underlying asset price in one day.

a. Calculate the first three moments of the change in the portfolio value.
b. Using the first two moments and assuming that the change in the portfolio is normally distributed, calculate the 1-day 95% VaR for the portfolio.

VaR with Fat Tails

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Student-*t*

- Now in stead of Normal distribution we consider student-*t* distribution with *v* degres of freedom.
- William Sealy Gosset wrote a paper in 1908 in *Biometrika* under the pseudonym "Student". Gosset worked at the Guinness Brewery.

Student-*t*

- The distribution is given by:

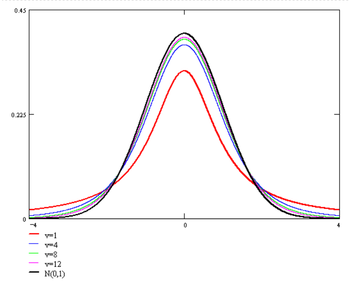
$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})} \left(1+\frac{t^2}{\nu}\right)^{-(\frac{\nu+1}{2})},$$

- Where Γ is the Gamma function given by:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t$$

Student-*t* Densities

- >>x=linspace(-4,4,1000);
- >>plot(tpdf(x,1))



Moments

- Mean=0, $v>1$
- Variance= $v/(v-2)$, $v>2$
- Skewness=0, $v>3$
- Kurtosis=$3+6/(v-4)$, $v>4$

VaR with Student-*t*

```
>>y=std(petr4).*trnd(10,1238,1).*sqrt(0.8)+mean(petr4)
>> qqplot(petr4,y)
```

Derive VaR with Student-*t*.

See tVaR.xls(VaR)

Back Testing

- This process is very importante. Tests how well VaR estimates would have performed in the past.
- Suppose we are calculating the 1-day 99% VaR. BackTesting results in observe how often the actual 1-day loss was greater than the 10-day 99% VaR calculated for that day.
- If that happens 1% of the days, we can feel conforable with the method used to compute VaR.
- But if for example we have 7% of the days we have evidence of an inadequate method.

Backtesting

>>VaR=(VaRSH1,VaRSH2,VaRNorm, VaRT-student)

>>kupiecbacktest(petro4,VaR(i),0.99)

Stochastic Volatility

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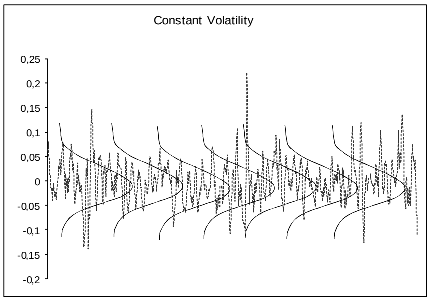
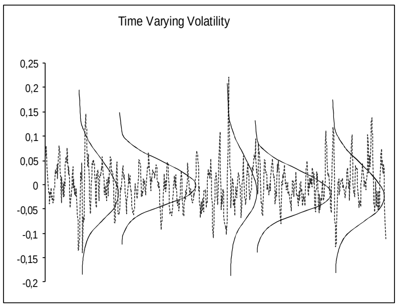
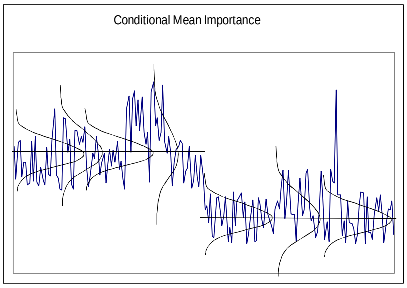
Volatility



Volatility



Conditional Mean



Standard Approach to Estimating Volatility

- Define σ_n as the volatility per day between day n-1 and day n, as estimated at end of day n-1
- Define S_i as the value of market variable at end of day i
- Define $u_i = \ln(S_i/S_{i-1})$

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \bar{u})^2$$

$$\bar{u} = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}$$

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Simplifications Usually Made in Risk Management

- Set $u_i = (S_i - S_{i-1})/S_{i-1}$
- Assume that the mean value of u_i is zero
- Replace $m-1$ by m

This gives $\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2$

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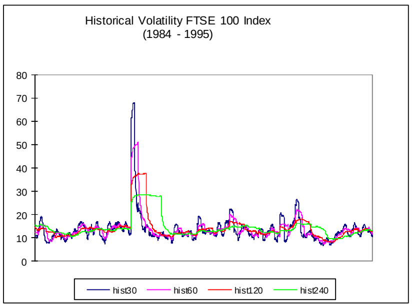
Problems



Weighting Scheme

Instead of assigning equal weights to the observations we can set

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\sum_{i=1}^{m} \alpha_i = 1$$

EWMA Model

- In an exponentially weighted moving average model, the weights assigned to the u^2 decline exponentially as we move back through time
- This leads to

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1-\lambda) u_{n-1}^2$$

To Show that Weights Decline Exponentially

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1-\lambda) u_{n-1}^2$$
$$= \lambda[\lambda \sigma_{n-2}^2 + (1-\lambda) u_{n-2}^2] + (1-\lambda) u_{n-1}^2$$
$$= (1-\lambda) u_{n-1}^2 + \lambda(1-\lambda) u_{n-2}^2 + \lambda^2 \sigma_{n-2}^2$$

Substituting for σ_{n-2}^2, then for σ_{n-3}^2 then for σ_{n-4}^2, and so on:

$$\sigma_n^2 = (1-\lambda) u_{n-1}^2 + \lambda(1-\lambda) u_{n-2}^2 + \lambda^2 (1-\lambda) u_{n-3}^2 +$$
$$\ldots + \lambda^{m-1}(1-\lambda) u_{n-m}^2 + \lambda^m \sigma_{n-m}^2$$

Weights start at 1 - λ and decline at rate λ

Attractions of EWMA

- Relatively little data needs to be stored
- We need only remember the current estimate of the variance rate and the most recent observation on the market variable
- Tracks volatility changes
- 0.94 is a popular choice for λ

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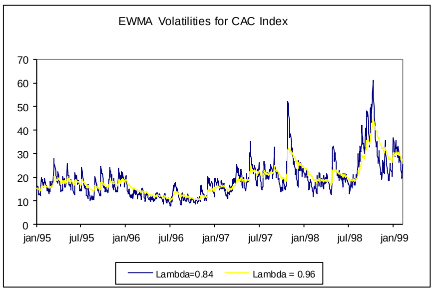
Example



Example: EWMA

- Assume λ=0.90 , na estimated volatility for day n-1 is 1% per day andduring day n-1 the Market variable has increased a 2%.
- That means σ_{n-1}^2 =0.01^2=0.0001and u_{n-1}=0.02^2=0.0004, from here:

σ_n^2 = 0.90x0.0001+0.10x0.0004=0.00013

That is σ_n is 1,14% per day.

RiskMetrics was developed by J.P.Morgan available since 1994, use λ=0.94, to up date volatility estimations.

ARCH(m)

Robert Engle: Nobel Prize 2003.

"Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation." Econometrica. Vol. 50, 1982, 987-1007.

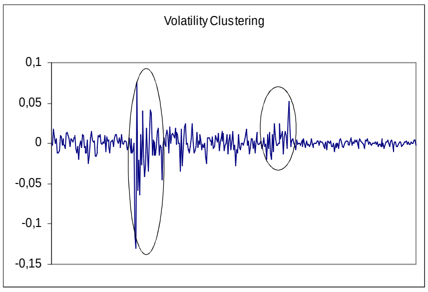
In a ARCH(m) model we assign weight to the long-run variance, V_L:

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

Conditional Autoregressive Variance



GARCH Model

- Bollerslev, T. "Generalized Autoregressive Conditional Heteroskedasticity." Journal of Econometrics. Vol. 31, 1986, 307-327
- GARCH(p,q):

$$\sigma^2_t = C + \sum_{i=1}^{p} \alpha_i u_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

GARCH (1,1)

In GARCH (1,1) we assign some weight to the long-run average variance rate

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

Since weights must sum to 1

$$\gamma + \alpha + \beta = 1$$

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GARCH (1,1) continued

Setting $\omega = \gamma V$ the GARCH (1,1) model is

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

and $V_L = \dfrac{\omega}{1-\alpha-\beta}$

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Example

- Suppose

$$\sigma_n^2 = 0.000002 + 0.13 u_{n-1}^2 + 0.86 \sigma_{n-1}^2$$

- The long-run variance rate is 0.0002 so that the long-run volatility per day is 1.4%

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Example continued

- Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.
- The new variance rate is

$$0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023336$$

The new volatility is 1.53% per day

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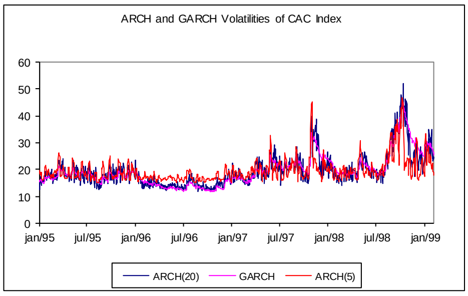
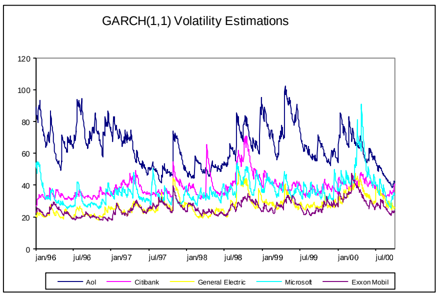
Examples



Examples



GARCH in Matlab & Eviews

- GARCH(1,1)

$$y_t = C + \varepsilon_t$$

$$\sigma_t^2 = \kappa + G_1\sigma_{t-1}^2 + A_1\varepsilon_{t-1}^2$$

- GARCH(p,q)

$$\sigma_t^2 = \kappa + \sum_{i=1}^{P} G_i\sigma_{t-i}^2 + \sum_{j=1}^{Q} A_j\varepsilon_{t-j}^2$$

GARCH(1,1)

```
Matlab:
>>[EstMdl,EstParamCov,logL,info]=estimate(garch(1,1),retpetr4a)
```

```
EVIEWS:
New/undated/1000/
import/excel
Quick/estimate/arch/retpetro4 c
```

Correlations and Covariances

Define $x_i=(X_i-X_{i-1})/X_{i-1}$ and $y_i=(Y_i-Y_{i-1})/Y_{i-1}$

Also

$\sigma_{x,n}$: daily vol of X calculated on day $n-1$

$\sigma_{y,n}$: daily vol of Y calculated on day $n-1$

cov_n: covariance calculated on day $n-1$

The correlation is $cov_n/(\sigma_{x,n}\sigma_{y,n})$

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Updating Correlations

- We can use similar models to those for volatilities
- Under EWMA

  $\text{cov}_n = \lambda\ \text{cov}_{n-1} + (1-\lambda)x_{n-1}y_{n-1}$

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Positive Finite Definite Condition

A variance-covariance matrix, $\mathbf{\Omega}$, is internally consistent if the positive semi-definite condition

$$\mathbf{w}^{\mathbf{T}}\mathbf{\Omega}\mathbf{w} \geq 0$$

for all vectors **w**

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Example

The variance-covariance matrix

$$\begin{pmatrix} 1 & 0 & 0.9 \\ 0 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{pmatrix}$$

is not internally consistent

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Volatilities and Correlations for Four-Index on Sept 25, 2008 with Equal Weights

	DJIA	FTSE	CAC 40	Nikkei 225
DJIA	1			
FTSE	0.489	1		
CAC 40	0.496	0.918	1	
Nikkei 225	−0.062	0.201	0.211	1

	DJIA	FTSE	CAC 40	Nikkei 225
Vol. per day (%)	1.11	1.42	1.40	1.38

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Volatilities and Correlations for Four-Index on Sept 25, 2008 for EWMA and λ=0.94

	DJIA	FTSE	CAC 40	Nikkei 225
DJIA	1			
FTSE	0.611	1		
CAC 40	0.629	0.971	1	
Nikkei 225	−0.113	0.409	0.342	1

	DJIA	FTSE	CAC 40	Nikkei 225
Vol. per day (%)	2.19	3.21	3.09	1.59

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One-Day 99% VaR Estimates

Historical Simulation	$253,385
Model Building Equal Weights	$217,757
Model Building EWMA	$471,025

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