Credit Risk
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Credit Ratings

- In the S&P rating system, AAA is the best rating. After that comes AA, A, BBB, BB, B, CCC, CC, and C
- The corresponding Moody's ratings are Aaa, Aa, A, Baa, Ba, B,Caa, Ca, and C
- Bonds with ratings of BBB (or Baa) and above are considered to be "investment grade"

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Estimating Default Probabilities

- Alternatives:
  - use historical data
  - use credit spreads
  - use Merton's model

Historical Data

Historical data provided by rating agencies are also used to estimate the probability of default

Cumulative Ave Default Rates (%)

(1970-2012, Moody's)

	1	2	3	4	5	7	10
Aaa	0.000	0.013	0.013	0.037	0.106	0.247	0.503
Aa	0.022	0.069	0.139	0.256	0.383	0.621	0.922
A	0.063	0.203	0.414	0.625	0.870	1.441	2.480
Baa	0.177	0.495	0.894	1.369	1.877	2.927	4.740
Ba	1.112	3.083	5.424	7.934	10.189	14.117	19.708
B	4.051	9.608	15.216	20.134	24.613	32.747	41.947
Caa-C	16.448	27.867	36.908	44.128	50.366	58.302	69.483

Interpretation

- The table shows the probability of default for companies starting with a particular credit rating
- A company with an initial credit rating of Baa has a probability of 0.177% of defaulting by the end of the first year, 0.495% by the end of the second year, and so on

Do Default Probabilities Increase with Time?

- For a company that starts with a good credit rating default probabilities tend to increase with time
- For a company that starts with a poor credit rating default probabilities tend to decrease with time

Conditional vs Unconditional Default Probabilities

- The conditional default probability is the probability of default for a certain time period conditional on no earlier default
- The unconditional default probability is the probability of default for a certain time period as seen at time zero
- What are the conditional and unconditional default probabilities for a Caa rated company in the third year?

Hazard Rate

- The hazard rate (also called default density), $\lambda(t)$, at time t is defined so that $\lambda(t)\Delta t$ is the conditional default probability for a short period between t and $t+\Delta t$
- If $V(t)$ is the cumulative probability of a company surviving to time t

$$V(t+\Delta t) - V(t) = -\lambda(t)V(t)\Delta t$$

This leads to

$$V(t) = e^{-\int_0^t \lambda(t)dt}$$

The cumulative probability of default by time t is

$$Q(t) = 1 - e^{-\bar{\lambda}(t)t}$$

Recovery Rate

- The recovery rate for a bond is usually defined as the price of the bond immediately after default as a percent of its face value
- Recovery rates tend to decrease as default rates increase

Recovery Rates; Moody's: 1982 to 2012

Class	Mean(%)
Senior Secured	51.6
Senior Unsecured	37.0
Senior Subordinated	30.9
Subordinated	31.5
Junior Subordinated	24.7

Using Credit Spreads

- Suppose $s(T)$ is the credit spread for maturity T
- Average hazard rate between time zero and time T is approximately

$$\frac{s(T)}{1-R}$$

where R is the recovery rate

- This estimate is very accurate in most situations

Explanation

- Loss rate at time t is $\lambda(t)(1-R)$
- If the credit spread is compensation for this loss rate it should approximately equal

$$\bar{\lambda}(t)(1-R)$$

Matching Bond Prices

- For more accuracy we can work forward in time choosing hazard rates that match bond prices
- This is another application of the bootstrap method

Real World vs Risk-Neutral Default Probabilities

- The default probabilities backed out of bond prices or credit default swap spreads are risk-neutral default probabilities
- The default probabilities backed out of historical data are real-world default probabilities

A Comparison

- Calculate 7-year default intensities from the Moody's data, 1970-2012, (These are real world default probabilities)
- Use Merrill Lynch data to estimate average 7-year default intensities from bond prices, 1996 to 2007 (these are risk-neutral default intensities)
- Assume a risk-free rate equal to the 7-year swap rate minus 10 basis points

Data from Moody's and Merrill Lynch

	Cumulative 7-year default probability (Moody's: 1970-2012)	Average bond yield spread in bps* (Merrill Lynch: 1996 to June 2007)
Aaa	0.247%	35.74
Aa	0.621%	43.67
A	1.441%	68.68
Baa	2.927%	127.53
Ba	14.117%	280.28
B	32.747%	481.04
Caa	58.302%	1103.70

*The benchmark risk-free rate for calculating spreads is assumed to be the swap rate minus 10 basis points. Bonds are corporate bonds with a life of approximately 7 years.

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Real World vs Risk Neutral Hazard Rates

Rating	Historical hazard rate[1] % per annum	Hazard rate from bond prices[2] (% per annum)	Ratio	Difference
Aaa	0.04	0.60	17.0	0.56
Aa	0.09	0.73	8.2	0.64
A	0.21	1.15	5.5	0.94
Baa	0.42	2.13	5.0	1.71
Ba	2.27	4.67	2.1	2.50
B	5.67	8.02	1.4	2.35
Caa	12.50	18.39	1.5	5.89

[1] Calculated as $-[\ln(1-d)]/7$ where d is the Moody's 7 yr default rate. For example, in the case of Aaa companies, d=0.00247 and -ln(0.99753)/7=0.0004 or 4bps. For investment grade companies the historical hazard rate is approximately d/7.

[2] Calculated as $s/(1-R)$ where s is the bond yield spread and R is the recovery rate (assumed to be 40%).

Average Risk Premiums Earned By Bond Traders

Rating	Bond Yield Spread over Treasuries (bps)	Spread of risk-free rate over Treasuries (bps)[1]	Spread to compensate for historical default rate (bps)[2]	Extra Risk Premium (bps)
Aaa	78	42	2	34
Aa	86	42	5	39
A	111	42	12	57
Baa	169	42	25	102
Ba	322	42	130	150
B	523	42	340	141
Caa	1146	42	750	323

[1] Equals average spread of our benchmark risk-free rate over Treasuries.

[2] Equals historical hazard rate times (1-R) where R is the recovery rate. For example, in the case of Baa, 25bps is 0.6 times 42bps.

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Possible Reasons for the Extra Risk Premium (The third reason is the most important)

- Corporate bonds are relatively illiquid
- The subjective default probabilities of bond traders may be much higher than the estimates from Moody's historical data
- **Bonds do not default independently of each other. This leads to systematic risk that cannot be diversified away.**
- Bond returns are highly skewed with limited upside. The non-systematic risk is difficult to diversify away and may be priced by the market

Which World Should We Use?

- We should use risk-neutral estimates for valuing credit derivatives and estimating the present value of the cost of default
- We should use real world estimates for calculating credit VaR and scenario analysis

Using Equity Prices: Merton's Model

- Merton's model regards the equity as an option on the assets of the firm
- In a simple situation the equity value is

$$\max(V_T - D, 0)$$

where V_T is the value of the firm and D is the debt repayment required

Equity vs. Assets

The Black-Scholes-Merton option pricing model enables the value of the firm's equity today, E_0, to be related to the value of its assets today, V_0, and the volatility of its assets, σ_V

$$E_0 = V_0 N(d_1) - De^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(V_0/D) + (r + \sigma_V^2/2)T}{\sigma_V \sqrt{T}}; \quad d_2 = d_1 - \sigma_V \sqrt{T}$$

Volatilities

$$\sigma_E E_0 = \frac{\partial E}{\partial V} \sigma_V V_0 = N(d_1) \sigma_V V_0$$

This equation together with the option pricing relationship enables V_0 and σ_V to be determined from E_0 and σ_E

Example

- A company's equity is $3 million and the volatility of the equity is 80%
- The risk-free rate is 5%, the debt is $10 million and time to debt maturity is 1 year
- Solving the two equations yields V_0=12.40 and σ_v=21.23%
- The probability of default is $N(-d_2)$ or 12.7%

CVA

- Credit value adjustment (CVA) is the amount by which a dealer must reduce the total value of transactions with a counterparty because of counterparty default risk

The CVA Calculation

Time	0	t_1	t_2	t_3	t_4	....	t_n=T
Default probability for counterparty		q_1	q_2	q_3	q_4	....	q_n
PV of expected loss given default		v_1	v_2	v_3	v_4	....	v_n

$$\text{CVA} = \sum_{i=1}^{n} q_i v_i$$

Calculation of q_i's

- Default probabilities are calculated from credit spreads

$$q_i = \exp\left(-\frac{s(t_{i-1})t_{i-1}}{1-R}\right) - \exp\left(-\frac{s(t_i)t_i}{1-R}\right)$$

Calculation of v_i's

- The v_i are calculated by simulating the market variables underlying the portfolio in a risk-neutral world
- If no collateral is posted the loss on a particular simulation trial during the *i*th interval is the PV of (1-R)max(V_i, 0) where V_i is the value of the portfolio at the mid point of the interval
- v_i is the average of the losses across all simulation trials

Two cases without MC simulation

Example 24.5

The Black–Scholes–Merton price of a 2-year uncollateralized option is \$3. Two-year zero-coupon bonds issued by the company selling the option have a yield 1.5% greater than the risk-free rate. The value of the option after default risk is considered is $3e^{-0.015\times 2}$ = \$2.91. (This assumes that the option stands alone and is not netted with other derivatives in the event of default.)

For the second special case, we consider a bank that has entered into an uncollateralized forward transaction with a counterparty where it has agreed to buy an asset for price K at time T. Define F_t as the forward price at time t for delivery of the asset at time T. The value of the transaction at time t is, from Section 5.7,

$$(F_t - K)e^{-r(T-t)}$$

where r is the risk-free interest rate (assumed constant).

Calculation of v_i

The bank's exposure at time t is therefore

$$\max[(F_t - K)e^{-r(T-t)}, 0] = e^{-r(T-t)}\max[F_t - K, 0]$$

The expected value of F_t in a risk-neutral world is F_0. The standard deviation of $\ln F_t$ is $\sigma\sqrt{t}$, where σ is the volatility of F_t. From equation (15A.1) the expected exposure at time t is therefore

$$w(t) = e^{-r(T-t)}\left[F_0 N(d_1(t)) - KN(d_2(t))\right]$$

where

$$d_1(t) = \frac{\ln(F_0/K) + \sigma^2 t/2)}{\sigma\sqrt{t}}, \quad d_2(t) = d_1(t) - \sigma\sqrt{t}$$

It follows that

$$v_i = w(t_i)e^{-rt_i}(1 - R)$$

15A.1

Key Result

If V is lognormally distributed and the standard deviation of $\ln V$ is w, then

$$E[\max(V - K, 0)] = E(V)N(d_1) - KN(d_2) \qquad \textbf{(15A.1)}$$

where

$$d_1 = \frac{\ln[E(V)/K] + w^2/2}{w}$$

$$d_2 = \frac{\ln[E(V)/K] - w^2/2}{w}$$

and E denotes the expected value.

Example

- 2 year forward. Current forward price is $1,600 per ounce. Two one-year intervals
- K = 1,500, σ = 20%, R = 0.3, r = 5%
- t_1 =0.5, t_2=1.5
- Suppose q_1 =0.02 and q_2=0.03
- v_1 = 92.67 and v_2 = 130.65
- CVA=0.02×92.67+0.03×130.65 = 5.77
- Value after CVA =
  (1600−1500)$e^{-0.05\times 2}$ – 5.77 = 84.71

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Example cont'

A bank has entered into a forward contract to buy 1 million ounces of gold from a mining company in 2 years for $1,500 per ounce. The current 2-year forward price is $1,600 per ounce. We suppose that only two intervals each 1-year long are considered in the calculation of CVA. The probability of the company defaulting during the first year is 2% and the probability that it will default during the second year is 3%. The risk-free rate is 5% per annum. A 30% recovery in the event of default is anticipated. The volatility of the forward price of gold is 20%.

In this case, $q_1 = 0.02$, $q_2 = 0.03$, $F_0 = 1{,}600$, $K = 1{,}500$, $\sigma = 0.2$, $r = 0.05$, $R = 0.3$, $t_1 = 0.5$, and $t_2 = 1.5$.

$$d_1(t_1) = \frac{\ln(1600/1500) + 0.2^2 \times 0.5}{0.2\sqrt{0.5}} = 0.5271$$

$$d_2(t_1) = d_1 - 0.2\sqrt{0.5} = 0.3856$$

so that

$$w(t_1) = e^{-0.05\times 1.5}[1600N(0.5271) - 1500N(0.3856)] = 135.73$$

and

$$v_1 = w(t_1)e^{-0.05\times 0.5} \times (1 - 0.3) = 92.67$$

Similarly $w(t_2) = 201.18$ and $v_2 = 130.65$.

The expected cost of defaults is

$$q_1 v_1 + q_2 v_2 = 0.02 \times 92.67 + 0.03 \times 130.65 = 5.77$$

The no-default value of the forward contract is $(1{,}600 - 1{,}500)e^{-0.05\times 2} = 90.48$.

Default Correlation

- The credit default correlation between two companies is a measure of their tendency to default at about the same time
- Default correlation is important in risk management when analyzing the benefits of credit risk diversification
- It is also important in the valuation of some credit derivatives, eg a first-to-default CDS and CDO tranches.

Measurement

- There is no generally accepted measure of default correlation
- Default correlation is a more complex phenomenon than the correlation between two random variables

Survival Time Correlation

- Define t_i as the time to default for company i and $Q_i(t_i)$ as the cumulative probability distribution for t_i
- The default correlation between companies i and j can be defined as the correlation between t_i and t_j
- But this does not uniquely define the joint probability distribution of default times

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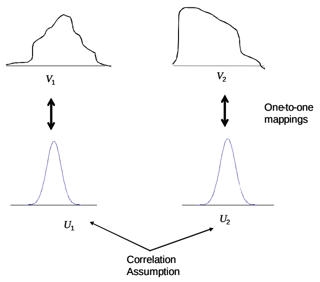
The Gaussian Copula Model



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Gaussian Copula Model

- Define a one-to-one correspondence between the time to default, t_i, of company i and a variable x_i by

$$Q_i(t_i) = N(x_i) \quad \text{or} \quad x_i = N^{-1}[Q(t_i)]$$

  where N is the cumulative normal distribution function.
- This is a "percentile to percentile" transformation. The p percentile point of the Q_j distribution is transformed to the p percentile point of the x_i distribution. x_i has a standard normal distribution
- We assume that the x_i are multivariate normal. The default correlation measure, ρ_{ij} between companies i and j is the correlation between x_i and x_j

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Example of Use of Gaussian Copula

Suppose that we wish to simulate the defaults for *n* companies . For each company the cumulative probabilities of default during the next 1, 2, 3, 4, and 5 years are 1%, 3%, 6%, 10%, and 15%, respectively

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Use of Gaussian Copula continued

- We sample from a multivariate normal distribution (with appropriate correlations) to get the x_i
- Critical values of x_i are

$N^{-1}(0.01) = -2.33$, $N^{-1}(0.03) = -1.88$,
$N^{-1}(0.06) = -1.55$, $N^{-1}(0.10) = -1.28$,
$N^{-1}(0.15) = -1.04$

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Use of Gaussian Copula continued

- When sample for a company is less than -2.33, the company defaults in the first year
- When sample is between -2.33 and -1.88, the company defaults in the second year
- When sample is between -1.88 and -1.55, the company defaults in the third year
- When sample is between -1,55 and -1.28, the company defaults in the fourth year
- When sample is between -1.28 and -1.04, the company defaults during the fifth year
- When sample is greater than -1.04, there is no default during the first five years

A One-Factor Model for the Correlation Structure

$$x_i = a_i F + \sqrt{1-a_i^2}\, Z_i$$

- The correlation between x_i and x_j is $a_i a_j$
- The ith company defaults by time T when $x_i < N^{-1}[Q_i(T)]$ or

$$Z_i < \frac{N^{-1}[Q_i(T) - a_i F]}{\sqrt{1-a_i^2}}$$

- Conditional on F, the probability of this is

$$Q_i(T|F) = N\left\{\frac{N^{-1}[Q_i(T)] - a_i F}{\sqrt{1-a_i^2}}\right\}$$

Credit VaR

- Can be defined analogously to Market Risk VaR
- A T-year credit VaR with an X% confidence is the loss level that we are X% confident will not be exceeded over T years

A rough estimate of the credit VaR when an X% confidence level is used and the time horizon is T is therefore $L(1-R)V(X,T)$, where L is the size of the loan portfolio and R is the recovery rate. The contribution of a particular loan of size L_i to the credit VaR is $L_i(1-R)V(X,T)$. This model underlies some of the formulas that regulators use for credit risk capital.[17]

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Calculation from a Factor-Based Gaussian Copula

- Consider a large portfolio of loans, each of which has a probability of $Q(T)$ of defaulting by time T. Suppose that all pairwise copula correlations are ρ so that all a_i's are $\sqrt{\rho}$
- We are X% certain that F is less than

$$N^{-1}(1-X) = -N^{-1}(X)$$

- It follows that the VaR is

$$V(X,T) = N\left\{\frac{N^{-1}[Q(T)] + \sqrt{\rho}\, N^{-1}(X)}{\sqrt{1-\rho}}\right\}$$

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Example

- A bank has $100 million of retail exposures
- 1-year probability of default averages 2% and the recovery rate averages 60%
- The copula correlation parameter is 0.1
- 99.9% worst case default rate is

$$V(0.999,1) = N\left(\frac{N^{-1}(0.02) + \sqrt{0.1}N^{-1}(0.999)}{\sqrt{1-0.1}}\right) = 0.128$$

- The one-year 99.9% credit VaR is therefore 100×0.128×(1-0.6) or $5.13 million

CreditMetrics

- Calculates credit VaR by considering possible rating transitions
- A Gaussian copula model is used to define the correlation between the ratings transitions of different companies