General Solution

$$L = \frac{1}{2}\sum_{a=1}^{A}\sum_{b=1}^{A} w_a w_b \sigma_{ab} - \lambda \cdot \left[\sum_{a=1}^{A} w_a \bar{r}_a - \bar{r}\right] - \mu \cdot \left[\sum_{a=1}^{A} w_a - 1\right]$$

Te optimal portfolio $\left[w_1^e, w_2^e, \ldots, w_A^e\right]$ c.p.o.:

$$\sum_{b=1}^{A} w_b^e \sigma_{ab} - \lambda \cdot \bar{r}_a - \mu = 0 \quad \forall\, a \geq 1$$

$$\sum_{a=1}^{A} w_a^e \bar{r}_a = \bar{r} \qquad e \qquad \sum_{a=1}^{A} w_a^e = 1$$

Solution for two assets

$$L = \frac{1}{2}\left(w_1^2\sigma_1^2 + w_1 w_2 \sigma_{12} + w_2 w_1 \sigma_{21} + w_2^2 \sigma_2^2\right) - \lambda \cdot \left[w_1 \bar{r}_1 + w_2 \bar{r}_2 - \bar{r}\right]$$

$$- \mu \cdot \left[w_1 + w_2 - 1\right]$$

Optimal solution $\left[w_1^e, w_2^e\right]$:

$$\frac{1}{2}\left(2 w_1^e \sigma_1^2 + w_2^e \sigma_{12} + w_2^e \sigma_{21}\right) - \lambda \cdot \bar{r}_1 - \mu = 0$$

$$\frac{1}{2}\left(w_1^e \sigma_{12} + w_1^e \sigma_{21} + 2 w_2^e \sigma_2^2\right) - \lambda \cdot \bar{r}_2 - \mu = 0$$

$$w_1^e \bar{r}_1 + w_2^e \bar{r}_2 = \bar{r} \qquad e \qquad w_1^e + w_2^e = 1$$

Tangent Portfolio (*P*)

How to find the tangent portfolio?

- The expected return of a portfolio which combines a risk free asset with a risky Q é dado pela linha reta:

$$E[r_C] = r_f + \frac{\left(E\lfloor r_Q \rfloor - r_f\right)}{\sigma_Q}\sigma_C \qquad ;$$

With trend:

$$S_Q = \frac{\left(E\lfloor r_Q \rfloor - r_f\right)}{\sigma_Q} \qquad .$$

- The investor chooses Q such that to maximize S.

Tangent Portfolio(P)

As:

$$S_Q = \frac{\sum_{a=1}^{A} w_a^Q \left(\bar{r}_a - r_f\right)}{\left(\sum_{a=1}^{A}\sum_{b=1}^{A} w_a^Q w_b^Q \sigma_{ab}\right)^{1/2}}$$

F.O.C. implies.:

$$\lambda \cdot \sum_{b=1}^{A} w_b \sigma_{ab} = \left(\bar{r}_a - r_f\right) \qquad \forall \quad a \geq 1$$

Doing $v_b = \lambda w_b$ we have:

$$\sum_{b=1}^{A} v_b \sigma_{ab} = \left(\bar{r}_a - r_f\right) \qquad \forall \quad a \geq 1 \qquad e \quad w_b = \frac{v_b}{\sum_{a=1}^{A} v_a}$$

Solution for A=2

$$w_1 = \frac{[ER_1 - r_f]\sigma_2^2 - [ER_2 - r_f]Cov(R_1, R_2)}{[ER_1 - r_f]\sigma_2^2 + [ER_2 - r_f]\sigma_1^2 - [ER_1 - r_f + ER_2 - r_f]Cov(R_1, R_2)}$$

$$w_2 = 1 - w_1$$